Pure spinor strings in TsT deformed background

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ABSTRACT: We consider pure spinor strings that propagate in the background generated by a sequence of TsT transformations. We use the fact that U(1) isometry variables of TsT-transformed background are related to the isometry variables of the initial background in the universal way that is independent of the details of the background. We will argue that after redefinitions of pure spinors and the fermionic variables we can construct pure spinor action with manifest U(1) isometry. This fact implies that the pure spinor string in TsT-transformed background is described by pure spinor string in the original background where world-volume modes are subject to twisted boundary conditions. We will argue that these twisted boundary conditions generally prevent to prove the quantum conformal invariance of the pure spinor string in $AdS_5 \times S^5$ background. We determine the conditions under which this quantum conformal invariance can be proved. We also determine the Lax pair for pure spinor strings in the TsT-transformed background.

KEYWORDS: Conformal Field Models in String Theory, AdS-CFT Correspondence.

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Contents

1. Introduction and summary

It was noticed recently in [1] that in situation when the initial geometry contains a two-torus, a regular background can be generated by using a combination of T-duality transformation on one angle variable followed a shift of the second isometry variable and finally performing the second T-duality along the first isometry variable. This chain of duality transformations that produces family of one-parameter deformation of initial background is known as TsT transformation. The work [1] can be generalised to construct regular multi-parameter deformations of gravity background if they contain a higher dimensional torus and it is possible to perform many chains of TsT transformations [2].

Remarkable fact considering TsT transformations is that they are very powerful for searching of new less supersymmetric examples of AdS/CFT correspondence. In particular, it was used in [1] to obtain a deformation of $AdS_5 \times S^5$ geometry that is conjectured to be dual to supersymmetric marginal deformation of $N = 4$ SYM. This deformation is called as a β deformation.

Some aspects of more general three-parameter deformed $AdS_5 \times S^5$ and the dual non-supersymmetric deformations of $N = 4$ SYM have been studied in papers [4–36]. It is unclear, however, if the non-supersymmetric string background is stable. For example, it is known that the spectrum of string theory in the TsT-transformed flat space contains tachyon [28]. However it does not imply that string theory on the deformed $AdS_5 \times S^5$ is unstable because the TsT-transformed flat space is singular at space infinity while the deformed $AdS_5 \times S^5$ is regular everywhere. As was shown in [2, 3] the TsT transformation has very nice property that it can be implemented on the string sigma model leading to the simple relations between string coordinates of the initial and TsT dual-transformed

background. These relations allow to prove that the classical solutions of string theory equations of motion in a deformed background are in one-to one correspondence with those in the initial background with twisted boundary conditions imposed on the $U(1)$ isometry fields that parametrise the torus.

The analysis performed in [2] was restricted to the bosonic part of type IIB Green-Schwarz superstring action on the deformed $AdS_5 \times S^5$. This work was generalised to the full Green-Schwarz superstring action in the remarkable paper [3]. The problem with superstring extension is how to define the TsT transformation for fermion variables since they are not neutral under T-duality transformations [73, 74]. The key idea that was presented in [3] and that solves this problem is to redefine the original fermions in such a way that they become neutral under the isometries of the torus.

The goal of this paper is to see that the same analysis can be performed in case of pure spinor string proposed by Berkovits [37 − 41].[1] In a recent paper [48], quantum consistency was argued by means of algebraic renormalization arguments. The one-loop conformal invariance of pure spinor string was also demonstrated in [50]. [2] Vertex operators for massless excitations have been proposed some time ago [47] and checked to be classically BRST invariant [55]. Algebra of currents was also classically calculated in [56] and the first attempt to calculate their operator product expansion was performed in [57].

All these results, especially proof of the quantum consistency of the pure spinor string in $AdS_5 \times S^5$ suggest that pure spinor string could be the correct way to study the string theory on the γ-deformed background. The goal of this paper is to demonstrate this fact. Let us outline its content.

We will show that we can formulate the pure spinor string in the deformed background using the TsT transformations from the original $AdS_5 \times S^5$ background. As in the case of GS superstring [3] we redefine both fermions and pure spinors variables in order that they become neutral under isometry transformations. Then we argue that the pure spinor string in γ-deformed $AdS_5 \times S^5$ background is equivalent to the pure spinor string in the original $AdS_5 \times S^5$ background where the world-sheet fields obey twisted boundary conditions. We will also argue that the existence of these twisted boundary conditions is crucial for the proof of the quantum consistency of the pure spinor string in γ deformed background. More precisely, the proof of the quantum consistency of pure spinor string in $AdS_5 \times S^5$ presented in [48] was based on the explicit gauge invariance of the pure spinor string in $AdS_5 \times S^5$ background. On the other hand the twisted boundary conditions for world-sheet fields are naturally related to the particular coset representative that however breaks the explicit gauge invariance of the theory. Then we will show that in order to restore this gauge invariance we have to restrict to the case when the world-sheet fields obey periodic boundary conditions.

Let us be more explicit. We will see that the configuration of the pure spinor string in $AdS_5 \times S^5$ background is labelled in general by 3 conserved angular momenta (J_1, J_2, J_3).

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[1]For review of pure spinor formalism in superstring theory, see [42 − 46].

[2]Check of the one-loop conformal invariance of pure spinor string in general background was performed in [51, 49].

These angular momenta depend on the deformation parameters γ_i through

$$\nu_i \equiv \epsilon_{ijk}\gamma_j J_k \ . \tag{1.1}$$

These combinations are the twists that appear in the relations between the angle variables of S^5 and the γ_i-deformed sphere. We will argue that for ν_i that are equal to integer (This situation occurs for rational γ_i only) and for $\nu_i = 0$ the currents of the pure spinor string in the $AdS_5 \times S^5$ background obey the periodic boundary conditions. Alternativelly, if the twists ν_i are not integer or not equal to zero then the twisted pure spinor string in the $AdS_5 \times S^5$ that are images of the closed string in the deformed geometry under inverse TsT transformations are open. On the other hand we will argue that strings with ν_i are equal to integer correspond to the string theory in $AdS_5 \times S^5$ background that pose the gauge invariance of the coset and that, according to the arguments by N. Berkovits given [48] has exact conformal invariance. Alternativelly, the twisted boundary conditions that arrise for non-integer ν_i correspond to the boundary conditions that break conformal invariance of the theory. This result confirms the analysis performed in [25].

Let us outline the structure of the paper. In next section (2) we review how the TsT transformation is defined in the context of the non-linear sigma model. In section (3) we introduce the action for pure spinor string in $AdS_5 \times S^5$ background. Then we determine its form using the explicit parametrisation of the coset introduced in [61]. In section (4) we study the equations of motions for pure spinor string in the coset representation. We prove the conservation of the BRST currents. In section (5) we perform the redefinition of the fermions and pure spinor variables following [3].

Then in section (6) we apply TsT transformation to the five sphere and we find the relation between the pure spinor string action in γ-deformed action and in the original $AdS_5 \times S^5$ action. Finally, in section (7) we argue for an existence of the Lax connection for pure spinor string in the γ deformed background again following the approach given in [3].

2. Review of the γ-deformed action

We start with the sigma model action that describes the propagation of closed string on the background with several U(1) isometries

$$S = -\frac{\sqrt{\lambda}}{4\pi} \int d\tau d\sigma \sqrt{-h}[h^{\mu\nu}\partial_\mu\phi^i\partial_\nu\phi^j G^0_{ij} - \epsilon^{\mu\nu}\partial_\mu\phi^i\partial_\nu\phi^j B^0_{ij} +$$
$$+2\partial_\mu\phi^i(h^{\mu\nu}U^0_{\nu,i} - \epsilon^{\mu\nu}V^0_{\nu,i}) + \mathcal{L}^0_{\text{rest}}] \ . \tag{2.1}$$

As usual we have introduced the effective string tension $\frac{\sqrt{\lambda}}{2\pi}$ that is identified with the 't Hooft coupling in the AdS/CFT correspondence, $h_{\mu\nu}$ is worldsheet metric with Minkowski signature that in conformal gauge is $h^{\mu\nu} = (-1, 1)$ and $\epsilon^{\mu\nu} = \frac{e^{\mu\nu}}{\sqrt{-h}}, e^{01} = -e^{10} = e^{\tau\sigma} = 1$. Next we assume that the action is invariant under the U(1) isometry transformations that are geometrically realised as shifts of the angle variables ϕ_i, $i = 1, 2, \ldots, d$. In other words the string background contains the d-dimensional torus T^d. The action (2.1) explicitly

shows the dependence on ϕ^i and their coupling to the background fields G^0_{ij}, B^0_{ij} and $U^0_{\nu,i}, V^0_{\nu,i}$. These background fields are independent on ϕ^i but can depend on other bosonic and fermionic string coordinates which are neutral under the U(1) isometry transformations. Finally $\mathcal{L}^0_{\text{rest}}$ denotes the part of the Lagrangian that depends on other fields of the theory.

As previous discussion suggests the action (2.1) is invariant under the constant shift of ϕ^i

$$\phi'^i(\tau, \sigma) = \phi^i(\tau, \sigma) + \epsilon^i . \tag{2.2}$$

The corresponding Noether currents have the form

$$J^\mu_i = -\frac{\sqrt{\lambda}}{2\pi}\sqrt{-h}(h^{\mu\nu}\partial_\nu\phi^j G^0_{ji} - \epsilon^{\mu\nu}\partial_\nu\phi^j B^0_{ij} + h^{\mu\nu}U^0_{\nu,i} - \epsilon^{\mu\nu}V^0_{\nu,i}) \tag{2.3}$$

and obeys the equation

$$\partial_\mu J^\mu_i = 0 \tag{2.4}$$

as a consequence of the equations of motion.

Now we are ready to study TsT duality of the angle variables. Let us consider two-torus that is generated by ϕ_1 and ϕ_2. The TsT transformation consists T-dualizing the variable ϕ_1 with the further shift $\phi_2 \rightarrow \phi_2 + \tilde{\gamma}\phi_1$ and dualizing ϕ_1 back. The TST transformation can be symbolically expressed as

$$(\phi_1, \phi_2) \overset{\text{TsT}}{\rightarrow} (\tilde{\phi}_1, \tilde{\phi}_2) . \tag{2.5}$$

In order to find the TsT transformation of the non-linear sigma model action we proceed following the classical works [71, 72].

Let us start with the T-duality on a circle parametrised by ϕ_1. As the next step we gauge the shift symmetry $\phi'^1 = \phi^1 + \epsilon^1$ so that ϵ^1 is now function of τ, σ. If we require that the action is invariant under the non-constant transformation we have to introduce the appropriate gauge field A_μ in such a way that

$$\partial_\mu\phi^1 \rightarrow (\partial_\mu\phi^1 + A_\mu) \equiv D_\mu\phi^1 . \tag{2.6}$$

At the same time we add to the action the term $\tilde{\phi}^1\epsilon^{\mu\nu}F_{\mu\nu}$ in order to assure that the gauge field has trivial dynamics. Then we obtain the gauge invariant action

$$S = -\frac{\sqrt{\lambda}}{4\pi}\int d\tau d\sigma\sqrt{-h}[h^{\mu\nu}D_\mu\phi^1 D_\nu\phi^1 G^0_{11} + 2h^{\mu\nu}D_\mu\phi^1\partial_\nu\phi^a G^0_{1a} + h^{\mu\nu}\partial_\mu\phi^a\partial_\nu\phi^b G^0_{ab} -$$
$$-\epsilon^{\mu\nu}\partial_\mu\phi^a\partial_\nu\phi^b B^0_{ab} - 2\epsilon^{\mu\nu}D_\mu\phi^1\partial_\nu\phi^b B^0_{1b} +$$
$$+2D_\mu\phi^1(h^{\mu\nu}U^0_{\nu,1} - \epsilon^{\mu\nu}V^0_{\nu,1}) + 2\partial_\mu\phi^a(h^{\mu\nu}U^0_{\nu,a} - \epsilon^{\mu\nu}V^0_{\nu,a}) + \tilde{\phi}^1\epsilon^{\mu\nu}F_{\mu\nu} + \mathcal{L}^0_{\text{rest}}], \tag{2.7}$$

where $a, b = 2, \dots, d$. Now thanks to the gauge invariance we can fix the gauge $\phi^1 = 0$ so that the action above takes the form

$$S = -\frac{\sqrt{\lambda}}{4\pi}\int d\tau d\sigma\sqrt{-h}[h^{\mu\nu}A_\mu A_\nu G^0_{11} + 2h^{\mu\nu}A_\mu\partial_\nu\phi^a G^0_{1a} + h^{\mu\nu}\partial_\mu\phi^a\partial_\nu\phi^b G^0_{ab} -$$
$$-\epsilon^{\mu\nu}\partial_\mu\phi^a\partial_\nu\phi^b B^0_{ab} - 2\epsilon^{\mu\nu}A_\mu\partial_\nu\phi^b B^0_{1b} +$$
$$+2A_\mu(h^{\mu\nu}U^0_{\nu,1} - \epsilon^{\mu\nu}V^0_{\nu,1}) + 2\partial_\mu\phi^a(h^{\mu\nu}U^0_{\nu,a} - \epsilon^{\mu\nu}V^0_{\nu,a}) + \tilde{\phi}^1\epsilon^{\mu\nu}F_{\mu\nu} + \mathcal{L}^0_{\text{rest}}] . \tag{2.8}$$

If we now integrate $\tilde{\phi}^1$ we obtain that $F_{\mu\nu} = 0$ and hence $A_\mu = \partial_\mu\theta$. Inserting back to the action (2.8) we obtain the original action (2.1) after identification $\theta = \phi^1$. On the other hand if we integrate out A_μ we obtain

$$2h^{\mu\nu}A_\nu G_{11}^0 + 2h^{\mu\nu}\partial_\nu\phi^a G_{1a}^0 - 2\epsilon^{\mu\nu}\partial_\nu\phi^a B_{1a}^0 + 2(h^{\mu\nu}U_{\nu,1}^0 - \epsilon^{\mu\nu}V_{\nu,1}) - 2\partial_\nu[\epsilon^{\nu\mu}\tilde{\phi}^1] = 0 \tag{2.9}$$

that implies

$$A_\mu = \frac{1}{G_{11}^0}(-\partial_\mu\phi^a G_{1a}^0 + h_{\mu\nu}\epsilon^{\nu\rho}\partial_\rho\phi^a B_{1a}^0 - (U_{\mu,1}^0 - h_{\mu\nu}\epsilon^{\nu\rho}V_{\rho,1}^0) - h_{\mu\nu}\epsilon^{\nu\rho}\partial_\rho\tilde{\phi}) \ . \tag{2.10}$$

Since we have argued that A_μ can be related to the original coordinate ϕ^1 as $A_\mu = \partial_\mu\phi^1$ the relation (2.10) implies following relation between ϕ^1 and $\tilde{\phi}^1$

$$\epsilon^{\nu\rho}\partial_\rho\tilde{\phi}^1 = -h^{\nu\rho}G_{11}^0\partial_\rho\phi^1 - h^{\nu\rho}\partial_\rho\phi^a G_{1a}^0 + \epsilon^{\nu\rho}\partial_\rho\phi^a B_{1a}^0 - h^{\nu\rho}U_{\rho,1}^0 + \epsilon^{\nu\rho}V_{\rho,1}^0 \ ,$$
$$\tilde{\phi}^a = \phi^a \ . \tag{2.11}$$

Now plugging the result (2.10) into the action above we obtain the action equivalent to (2.1)

$$S = -\frac{\sqrt{\lambda}}{4\pi}\int d\tau d\sigma \sqrt{-h}[h^{\mu\nu}\partial_\mu\tilde{\phi}^i\partial_\nu\tilde{\phi}^j\tilde{G}_{ij} - \epsilon^{\mu\nu}\partial_\mu\tilde{\phi}^i\partial_\nu\tilde{\phi}^j\tilde{B}_{ij} +$$
$$+2\partial_\mu\phi^i(h^{\mu\nu}\tilde{U}_{\nu,i} - \epsilon^{\mu\nu}\tilde{V}_{\nu,i}) + \tilde{\mathcal{L}}_{\text{rest}}] \ , \tag{2.12}$$

where now

$$\tilde{G}_{11} = \frac{1}{G_{11}^0} \ , \qquad \tilde{G}_{ab} = G_{ab}^0 - \frac{G_{a1}^0 G_{1b}^0 - B_{1a}^0 B_{1b}^0}{G_{11}^0} \ ,$$
$$\tilde{G}_{1a} = \frac{B_{1a}^0}{G_{11}^0} \ , \qquad \tilde{B}_{ab} = B_{ab}^0 - \frac{G_{1a}^0 B_{1b}^0 - B_{1a}^0 G_{1b}^0}{G_{11}^0} \ ,$$
$$\tilde{B}_{1a} = \frac{G_{1a}^0}{G_{11}^0} \ , \qquad \tilde{B}_{a1} = -\frac{G_{1a}^0}{G_{11}^0} \ ,$$
$$\tilde{U}_{\mu,1} = \frac{V_{\mu,1}^0}{G_{11}^0} \ , \qquad \tilde{V}_{\mu,1} = \frac{U_{\mu,1}^0}{G_{11}^0} \ ,$$
$$\tilde{U}_{\mu,a} = U_{\mu,a}^0 - \frac{G_{1,a}^0 U_{\nu,1}^0 - B_{1a}^0 V_{\mu,1}^0}{G_{11}^0} \ ,$$
$$\tilde{V}_{\mu,a} = V_{\mu,a}^0 - \frac{G_{1a}^0 V_{\mu,1}^0 - B_{1a}^0 U_{\mu,1}^0}{G_{11}^0} \ ,$$
$$\tilde{\mathcal{L}}_{\text{rest}} = \mathcal{L}_{\text{rest}}^0 - h^{\mu\nu}\frac{U_{\mu,1}^0 U_{\nu,1}^0 - V_{\mu,1}^0 V_{\nu,1}^0}{G_{11}^0} + \epsilon^{\mu\nu}\frac{U_{\mu,1}^0 V_{\nu,1}^0 - V_{\mu,1}^0 U_{\nu,1}^0}{G_{11}^0} \ .$$

Clearly the action (2.12) has the same number of symmetries as the original one.

The next step in the definition of the TsT transformation is the the shift of the variables $\tilde{\phi}^a$ that is defined as

$$\tilde{\phi}^2 = \tilde{\phi}_s^2 + \hat{\gamma}\tilde{\phi}_s^1, \quad \tilde{\phi}_s^1 = \tilde{\phi}^1 \ ,$$
$$\tilde{\phi}_s^a = \tilde{\phi}^a \ , \quad a = 3,\ldots,d \ . \tag{2.13}$$

If we now insert (2.13) into the action (2.12) we get

$$S = -\frac{\sqrt{\lambda}}{4\pi} \int d\tau d\sigma \sqrt{-h} [h^{\mu\nu} \partial_\mu \tilde{\phi}_s^i \partial_\nu \tilde{\phi}_s^j \tilde{G}_{ij}^s - \epsilon^{\mu\nu} \partial_\mu \tilde{\phi}_s^i \partial_\nu \tilde{\phi}_s^j \tilde{B}_{ij}^s +$$
$$+2\partial_\mu \tilde{\phi}_s^i (h^{\mu\nu} \tilde{U}_{\nu,i}^s - \epsilon^{\mu\nu} \tilde{V}_{\nu,i}^s) + \tilde{\mathcal{L}}_{\rm rest}] , \tag{2.14}$$

where the forms of the background fields $\tilde{G}_{ij}^s$, $\tilde{B}_{ij}^s$, $\tilde{U}_{\nu,i}^s$ and $\tilde{V}_{\nu,i}^s$ can be easily determined from the action (2.12) and the shift transformation (2.13).

Finally we perform the last T-duality transformation along the direction labelled with $\tilde{\phi}_s^1$. After this transformation we get the action in the final form

$$S = -\frac{\sqrt{\lambda}}{4\pi} \int d\tau d\sigma \sqrt{-h} [h^{\mu\nu} \partial_\mu \phi_F^i \partial_\nu \phi_F^j G_{ij} - \epsilon^{\mu\nu} \partial_\mu \phi_F^i \partial_\nu \phi_F^j B_{ij} +$$
$$+2\partial_\mu \phi_F^i (h^{\mu\nu} U_{\nu,i} - \epsilon^{\mu\nu} V_{\nu,i}) + \mathcal{L}_{\rm rest}] , \tag{2.15}$$

where now

$$G_{ij} = \frac{G_{ij}^0}{D} ,$$
$$G_{ia} = G_{ai} = \frac{G_{ia}^0}{D} + \hat{\gamma} \frac{B_{2a}^0 G_{1i}^0 - B_{1a}^0 G_{2i}^0 + B_{12}^0 G_{ia}^0}{D} ,$$
$$G_{ab} = G_{ab}^0 + \frac{\hat{\gamma} + \hat{\gamma}^2 B_{12}^0}{D} 2(B_{23}^0 G_{13}^0 - B_{13}^0 G_{23}^0) +$$
$$+\frac{\hat{\gamma}^2}{D} \left(G_{11}^0 (B_{2a}^0 B_{2b}^0 - G_{2a}^0 G_{2b}^0 + G_{22}^0 (B_{1a}^0 B_{1b}^0 - G_{1a}^0 G_{1b}^0) + 2G_{12}^0 (G_{2a}^0 G_{1b}^0 - B_{1a}^0 B_{2b}^0) \right) \tag{2.16}$$

and

$$B_{12} = -B_{21} = \frac{B_{12}^0}{D} + \frac{\hat{\gamma}}{D} (G_{11}^0 G_{22}^0 - (G_{12}^0)^2 + (B_{12}^0)^2) ,$$
$$B_{ia} = -B_{ai} = \frac{B_{ia}^0}{D} + \frac{\hat{\gamma}}{D} (G_{2a}^0 G_{i1}^0 - G_{i2}^0 G_{1a}^0 + B_{12}^0 B_{ia}^0) ,$$
$$U_{\mu,i} = \frac{U_{\mu,i}^0}{D} + \frac{\hat{\gamma}}{D} (G_{11}^0 V_{\mu,2}^0 - G_{2i}^0 V_{\mu,1}^0 + B_{12}^0 U_{\mu,i}^0) ,$$
$$V_{\mu,i} = \frac{V_{\mu,i}^0}{D} + \frac{\hat{\gamma}}{D} (B_{12}^0 V_{\mu,i}^0 + G_{1i}^0 U_{\mu,2}^0 - G_{2i}^0 U_{\mu,1}^0) ,$$
$$U_{\mu,a} = U_{\mu,a}^0 + \frac{(\hat{\gamma} + \hat{\gamma}^2 B_{12}^0)}{D} (\epsilon^{ij} G_{ia}^0 V_{\mu,j}^0 - \epsilon^{ij} B_{ia}^0 U_{\mu,j}^0) +$$
$$+\frac{\hat{\gamma}^2}{D} (\epsilon^{ij} U_{\mu,i}^0 (G_{2a}^0 G_{1j}^0 - G_{1a}^0 G_{2j}^0) + \epsilon^{ij} V_{\mu,i}^0 (-B_{2a}^0 G_{1j}^0 + B_{1a}^0 G_{2j}^0)) ,$$
$$V_{\mu,a} = V_{\mu,a}^0 + \frac{(\hat{\gamma} + \hat{\gamma}^2 B_{12}^0)}{D} (\epsilon^{ij} G_{i3}^0 U_{\mu,j}^0 - \epsilon^{ij} B_{ia}^0 V_{\mu,j}^0) +$$
$$+\frac{\hat{\gamma}^2}{D} (\epsilon^{ij} V_{\mu,i}^0 (G_{2a}^0 G_{1j}^0 - G_{1a}^0 G_{2j}^0) + \epsilon^{ij} U_{\mu,i}^0 (-B_{2a}^0 G_{1j}^0 + B_{1a}^0 G_{2j}^0)) ,$$
$$\mathcal{L}_{\rm rest} = \mathcal{L}_{\rm rest}^0 + \frac{(\hat{\gamma} + \hat{\gamma}^2 B_{12}^0)}{D} (2\epsilon^{ij} (V_{0,i}^0 V_{1,j}^0 - U_{0,i}^0 U_{1,j}^0 + h^{\mu\nu} U_{\mu,i}^0 V_{\nu,j}^0)) +$$
$$+\frac{\hat{\gamma}^2}{D} (G_{ij}^0 \epsilon^{i\bar{i}} \epsilon^{j\bar{j}} h^{\mu\nu} (V_{\mu,\bar{i}}^0 V_{\nu,\bar{j}}^0 - U_{\mu,\bar{i}}^0 U_{\nu,\bar{j}}^0) + G_{ij}^0 \epsilon^{i\bar{i}} \epsilon^{j\bar{j}} \epsilon^{\mu\nu} U_{\mu\bar{i}}^0 V_{\nu\bar{j}}^0) , \tag{2.17}$$

where $i, j = 1, 2$ define the directions of a two-torus and the index a runs over $3, \ldots, d$. The element D is given by

$$D = 1 + 2\hat{\gamma}G_{12}^0 + \hat{\gamma}^2(G_{11}^0 G_{22}^0 - (G_{12}^0)^2 + (B_{12}^0)^2), \quad \hat{\gamma} = \sqrt{\lambda}\gamma. \quad (2.18)$$

Repeating the arguments given below the first T-duality transformation we can find the relation between between ϕ_F and ϕ in the form

$$\partial_\mu \phi_F^1 = \partial_1 \phi^1 - \hat{\gamma}\epsilon_{\mu\nu}h^{\nu\rho}\partial_\rho \phi^i G_{i2} + \hat{\gamma}\partial_\mu \phi^i B_{i2} - \hat{\gamma}\epsilon_{\mu\nu}h^{\nu\rho}U_{\rho 2} - \hat{\gamma}V_{\mu 2},$$
$$\partial_\mu \phi_F^2 = \partial_\mu \phi^2 + \hat{\gamma}\epsilon_{\mu\nu}h^{\nu\rho}\partial_\rho \phi^i G_{i1} - \hat{\gamma}\partial_\mu \phi^i B_{i1} + \hat{\gamma}\epsilon_{\mu\nu}h^{\nu\rho}U_{\rho 1} + \hat{\gamma}V_{\mu 1}, \quad i, j = 1, \ldots d,$$
$$\partial_\mu \phi_F^a = \partial_\mu \phi^a, a = 3, \ldots, d. \quad (2.19)$$

In what follows we rename ϕ^F as $\tilde{\phi}$ in order to have contact with [3]. Clearly the action (2.15) has the same number of symmetries as related to the the constant shifts of the variables $\tilde{\phi}$. The conserved Noether currents have the form

$$\tilde{J}_i^\mu = -\frac{\sqrt{\lambda}}{2\pi}\sqrt{-h}(h^{\mu\nu}\partial_\nu \tilde{\phi}^j G_{ji} - \epsilon^{\mu\nu}\partial_\nu \tilde{\phi}^j B_{ij} + h^{\mu\nu}U_{\nu,i} - \epsilon^{\mu\nu}V_{\nu,i}). \quad (2.20)$$

It is important to stress that following relations holds [2, 3]

$$\tilde{J}_i^\mu(\tilde{\phi}) = J_i^\mu(\phi). \quad (2.21)$$

Now using (2.19), (2.21) together with (2.3) and (2.20) we obtain

$$\partial_1 \tilde{\phi}^1 - \partial_1 \phi^1 = -\gamma J_2^\tau,$$
$$\partial_1 \tilde{\phi}^2 - \partial_1 \phi^1 = \gamma J_1^\tau,$$
$$\partial_1 \tilde{\phi}^i - \partial_1 \phi^i = 0, \quad i > 2. \quad (2.22)$$

Since we consider the closed string on the γ-deformed background the angle variables $\tilde{\phi}^i$ have to have following periodicity conditions

$$\tilde{\phi}^i(2\pi) - \tilde{\phi}^i(0) = 2\pi n_i, n_i \in Z. \quad (2.23)$$

Then integrating (2.22) we obtain the relation between the original variables

$$\phi^1(2\pi) - \phi^1(0) = 2\pi(n_1 + \gamma J_2),$$
$$\phi^2(2\pi) - \phi^2(0) = 2\pi(n_2 - \gamma J_1), \quad (2.24)$$

where

$$J_i = \frac{1}{2\pi}\int_0^{2\pi} d\sigma J_i^\tau(\sigma), \quad (2.25)$$

and where J_i are constant as follows from (2.4).

Now we can also look on this problem from another point of view using the fact that the momentum conjugate to ϕ^i coincides with J_i^τ. Therefore we can rewrite the time component of (2.22) in the form

$$\tilde{p}_i = p_i, \quad \partial_\sigma \tilde{\phi}^i - \partial_\sigma \phi^i = -\gamma_{ij}p^j, \quad i, j = 1, \ldots, d, \quad (2.26)$$

where $\gamma_{ij} = -\gamma_{ji}$ with one nonzero component $\gamma_{12} \equiv \gamma$. It is clear that (2.26) up to twisted boundary conditions a TsT transformation is just a simple linear canonical transformation of the $\mathrm{U}(1)$ isometry variables. Then the twist is the origin of the nonequivalence of the original and transformed theories. It is also clear that the most general multi-parameter TsT transformed background obtained by applying TsT transformations successively, many times when each time we pick up a new torus and a new deformation parameter is completely parametrised by the relation (2.26) with arbitrary matrix γ_{ij}. Therefore background that contains d dimensional torus admits $d(d-1)/2$ -parameter TsT transformation. In case of AdS_5 the most general TsT transformation applies to the five sphere S^5 (In order to preserve an isometry of AdS_5) has three independent parameters. The twisted boundary conditions for the original angles ϕ^i in the case of the most general deformation take the form

$$\phi^i(2\pi) - \phi^i(0) = 2\pi(n^i - \nu^i) \,, \quad \nu_i = -\gamma_{ik}J_k \,. \tag{2.27}$$

The general three-parameter γ-deformed background is obtained by applying the TsT transformation three times. Following [3] we express the corresponding procedure as

$$(\phi_1, \phi_2, \phi_3) \xrightarrow{\gamma_3} (\tilde{\phi}_1, \tilde{\phi}_2, \tilde{\phi}_3) \xrightarrow{\gamma_1} (\tilde{\tilde{\phi}}_1, \tilde{\tilde{\phi}}_2, \tilde{\tilde{\phi}}_3) \xrightarrow{\gamma_2} (\check{\phi}_1, \check{\phi}_2, \check{\phi}_3) \,. \tag{2.28}$$

Since under every step the corresponding Noether currents remain the same we can write

$$
\begin{array}{lll}
\tilde{\phi}'_1 - \phi'_1 = -\gamma_3 J_2^\tau & \tilde{\tilde{\phi}}'_1 - \tilde{\phi}'_1 = 0 & \check{\phi}_1 - \tilde{\tilde{\phi}}'_1 = \gamma_2 J_3^\tau \\
\tilde{\phi}'_2 - \phi'_2 = \gamma_3 J_1^\tau & \tilde{\tilde{\phi}}'_2 - \tilde{\phi}'_2 = -\gamma_1 J_3^\tau & \check{\phi}'_2 - \tilde{\tilde{\phi}}_2 = 0 \\
\tilde{\phi}'_3 - \phi'_3 = 0 & \tilde{\tilde{\phi}}_3 - \tilde{\phi}'_3 = \gamma_1 J_2^\tau & \check{\phi}'_3 - \tilde{\tilde{\phi}}_3 = -\gamma_2 J_1^\tau
\end{array}
\tag{2.29}
$$

From these formula's we can find the relation between ϕ^i and $\tilde{\phi}^i$ in the form

$$\partial_\sigma \tilde{\phi}^i - \partial_\sigma \phi^i \epsilon_{ijk} \gamma_j J_k^\tau \,, \quad \gamma_{ik} = -\epsilon_{ijk}\gamma_j \,. \tag{2.30}$$

Integrating this equation and using the fact that $\tilde{\phi}^i(2\pi) - \tilde{\phi}^i(0) = 2\pi n^i$ we obtain the twisted boundary conditions for the original angles

$$\phi^i(2\pi) - \phi^i(0) = 2\pi(n^i - \nu^i) \,, \quad \nu^i = \epsilon_{ijk}\gamma_j J_k \,. \tag{2.31}$$

3. Pure spinor action in $AdS_5 \times S^5$ and explicit coset representation

The pure spinor action in $AdS_5 \times S^5$ was introduced in [47, 48] and further studied in [55, 58]. In the covariant worldsheet description the pure spinor string action on $AdS_5 \times S^5$ takes the form

$$
\begin{aligned}
S = {} & -\frac{\sqrt{\lambda}}{2\pi} \int d\tau d\sigma \sqrt{-\eta} \, \mathrm{Str}[\frac{1}{2}\eta^{\mu\nu} \left(J_\mu^{(2)} J_\nu^{(2)} + J_\mu^{(1)} J_\nu^{(3)} + J_\mu^{(3)} J_\nu^{(1)} \right) + \\
& + \frac{\epsilon^{\mu\nu}}{4} \left(J_\mu^{(1)} J_\nu^{(3)} - J_\mu^{(3)} J_\nu^{(1)} \right)] + S_{\mathrm{ghost}} \,, \\
S_{\mathrm{ghost}} = {} & -\frac{\sqrt{\lambda}}{2\pi} \int d\tau d\sigma \sqrt{-\eta} \, \mathrm{Str}[w_\mu \mathcal{P}^{\mu\nu} \partial_\nu \lambda + \hat{w}_\mu \tilde{\mathcal{P}}^{\mu\nu} \partial_\nu \hat{\lambda} + \\
& + N_\mu \mathcal{P}^{\mu\nu} J_\nu^{(0)} + \hat{N}_\mu \tilde{\mathcal{P}}^{\mu\nu} J_\nu^{(0)} - \frac{1}{2} N_\mu \mathcal{P}^{\mu\nu} \hat{N}_\nu - \frac{1}{2} \hat{N}_\mu \tilde{\mathcal{P}}^{\mu\nu} N_\nu] \,,
\end{aligned}
\tag{3.1}
$$

where we have introduced the notation

$$
\begin{aligned}
J_\mu^{(0)} &= (g^{-1}\partial_\mu g)^{[\underline{cd}]} T_{[\underline{cd}]} \,, & J_\mu^{(1)} &= (g^{-1}\partial_\mu g)^\alpha T_\alpha \,, \\
J_\mu^{(2)} &= (g^{-1}\partial_\mu g)^{\underline{c}} T_{\underline{c}} \,, & J_\mu^{(3)} &= (g^{-1}\partial_\mu g)^{\hat\alpha} T_{\hat\alpha} \,, \\
w_\mu &= w_{\mu\alpha} T_{\hat\alpha} \delta^{\alpha\hat\alpha} \,, & \lambda &= \lambda^\alpha T_\alpha \,, \\
N_\mu &= -\{w_\mu, \lambda\} - N_\mu^{cd} T_{[cd]} + N_\mu^{c'd'} T_{[c'd']} \,, \\
\hat w_\mu &= \hat w_{\mu\hat\alpha} T_\alpha \delta^{\hat\alpha\alpha} \,, & \hat\lambda &= \hat\lambda^{\hat\alpha} T_{\hat\alpha} \,, \\
\hat N_\mu &= -\left\{\hat w_\mu, \hat\lambda\right\} = -\hat N_\mu^{cd} T_{[cd]} + \hat N_\mu^{c'd'} T_{[c'd']} \,.
\end{aligned}
\tag{3.2}
$$

We also work with the flat worldsheet metric where $h_{\mu\nu} = \eta_{\mu\nu} = \mathrm{diag}(-1,1)$ and where we have also defined

$$
\mathcal{P}^{\mu\nu} = (\eta^{\mu\nu} - \epsilon^{\mu\nu}) \,, \quad \tilde{\mathcal{P}}^{\mu\nu} = (\eta^{\mu\nu} + \epsilon^{\mu\nu}) \,, \quad \epsilon^{\mu\nu} \frac{e^{\mu\nu}}{\sqrt{-\eta}} \,, \quad e^{01} = -e^{10} = 1 \,.
\tag{3.3}
$$

In what follows we work in coordinates $x^0 = \tau, x^1 = \sigma$ where $\sigma \in (0, 2\pi)$.

An element M of the superalgebra $\mathbf{su}(2,2|4)$ is given by a 8×8 matrix that can be written in terms of 4×4 blocks as

$$
M = \begin{pmatrix} A & X \\ Y & D \end{pmatrix} \,.
\tag{3.4}
$$

The superalgebra $\mathbf{su}(2,2|4)$ is singled out by requiring that M has to have zero supertrace $\mathrm{Str} M = \mathrm{Tr} A - \mathrm{Tr} D = 0$ and to satisfy the following reality condition

$$
HM + M^\dagger H = 0 \,.
\tag{3.5}
$$

The choice of the hermitian matrix H is not unique and we choose H to be of the diagonal form

$$
H = \begin{pmatrix} \Sigma & 0 \\ 0 & 1 \end{pmatrix} \,.
\tag{3.6}
$$

Then (3.5) and (3.6) imply

$$
D = -D^\dagger, \quad \Sigma A = -A^\dagger \Sigma, \quad Y = -X^\dagger \Sigma \,,
\tag{3.7}
$$

where

$$
\Sigma = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \,.
\tag{3.8}
$$

The algebra $\mathbf{su}(2,2|4)$ also contains the $\mathbf{u}(1)$ generator $i\mathbf{I}$ where $\mathbf{I}$ is identity matrix of the corresponding dimension. The superalgebra $\mathbf{psu}(2,2|4)$ is defined as the quotient algebra of $\mathbf{su}(2,2|4)$ over this $\mathbf{u}(1)$ factor; it has no realisation in terms of 8×8 matrices.

The essential feature of the superalgebra $\mathbf{su}(2,2|4)$ is that it admits a $\mathbf{Z}_4$ automorphism Ω such that the condition $\Omega(H) = H$ determines the maximal subgroup to be $\mathrm{SO}(4,1) \times$

SO(5) that leads to the definition of the coset for the sigma model. The $\mathbf{Z}_4$ automorphism Ω takes an element of $\mathbf{psu}(2,2|4)$ to another $G \to \Omega(G)$ such that

$$\Omega(G) = \begin{pmatrix} KA^T K & -KY^T K \\ KX^T K & KB^T K \end{pmatrix}, \quad K = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}. \tag{3.9}$$

Since $\Omega^4 = 1$ the eigenvalues of Ω are i^p, $p = 0, 1, 2, 3$. Therefore we can decompose the superalgebra G as

$$G = \mathcal{H}_0 \oplus \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \mathcal{H}_3, \tag{3.10}$$

where $\mathcal{H}_p$ denotes the eigenspace of Ω such that if $H \in \mathcal{H}_p$ then

$$\Omega(H) = i^p H. \tag{3.11}$$

Explicitly, any matrix M of $\mathbf{su}(2,2|4)$ can be decomposed into the sum

$$M = M^{(0)} + M^{(1)} + M^{(2)} + M^{(3)}, \tag{3.12}$$

where

$$M^{(0)} = \frac{1}{4}(M + \Omega(M) + \Omega^2(M) + \Omega^3(M)) = \frac{1}{2}\begin{pmatrix} A + KA^T K & 0 \\ 0 & D + KD^T K \end{pmatrix},$$

$$M^{(2)} = \frac{1}{4}(M - \Omega(M) + \Omega^2(M) - \Omega^3(M)) = \frac{1}{2}\begin{pmatrix} A - KA^T K & 0 \\ 0 & D - KD^T K \end{pmatrix},$$

$$M^{(1)} = \frac{1}{4}(M - i\Omega(M) - \Omega^2(M) + i\Omega^3(M)) = \frac{1}{2}\begin{pmatrix} 0 & X + iKY^T K \\ Y - iKX^T K & 0 \end{pmatrix},$$

$$M^{(3)} = \frac{1}{4}(M + i\Omega(M) - \Omega^2(M) - i\Omega^3(M)) = \frac{1}{2}\begin{pmatrix} 0 & X - iKY^T K \\ Y + iKX^T K & 0 \end{pmatrix},$$
$$\tag{3.13}$$

and where $\Omega(M^{(p)}) = i^p M^{(p)}$. We see that $M^{(0)}$ form $\mathbf{so}(4,1) \times \mathbf{so}(5)$ subalgebra which we wish to mod out in the coset. We also see that the matrices $M^{(1,3)}$ contain the odd matrices. Splitting M into Grassman even and odd parts

$$M = M_{\text{even}} + M_{\text{odd}}, \quad M_{\text{even}} = \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}, \quad M_{\text{odd}} = \begin{pmatrix} 0 & X \\ Y & 0 \end{pmatrix}, \tag{3.14}$$

we can rewrite the expressions for $M^{(p)}$ in the following form

$$M^{(0)} = \frac{1}{2}(M_{\text{even}} + K_8 M_{\text{even}}^T K_8), \quad M^{(2)} = \frac{1}{2}(M_{\text{even}} - K_8 M_{\text{even}}^T K_8),$$

$$M^{(1)} = \frac{1}{2}(M_{\text{odd}} + i\tilde{K}_8 M_{\text{odd}}^T K_8), \quad M^{(3)} = \frac{1}{2}(M_{\text{odd}} - i\tilde{K}_8 M_{\text{odd}}^T K_8), \tag{3.15}$$

where K_8 and $\tilde{K}_8$ are defined as

$$K_8 = \begin{pmatrix} K & 0 \\ 0 & K \end{pmatrix}, \quad \tilde{K}_8 = \begin{pmatrix} K & 0 \\ 0 & -K \end{pmatrix}.$$ (3.16)

The next step is to explicit choose the coset representative g. Following [3] we take the coset parametrisation in the form

$$g = g(\theta)g(z).$$ (3.17)

Here $g(\theta) = \exp(\theta)$, where θ is an element of $\mathbf{psu}(2,2|4)$ that contains 32 fermionic degrees of freedom. The element $g(z)$ belongs to $SU(2,2) \times SU(4)$ and takes following form [61]

$$g(z) = \begin{pmatrix} \tilde{g}_a(x) & 0 \\ 0 & \tilde{g}_s(y) \end{pmatrix},$$ (3.18)

where

$$\tilde{g}_a(x) = \exp\frac{1}{2}(x_a\gamma_a), \quad \tilde{g}_s(y) = \exp\frac{i}{2}(y_a\Gamma_a),$$ (3.19)

where $z \equiv (x_a, y_a)$ and x_a parametrise the AdS_5 space while y_a stand for the five-sphere. The matrices $\Gamma_a, \gamma_a, a = 1, \ldots, 5$ are Dirac matrices for $SO(5)$ and $SO(4,1)$ respectively. These matrices obey the relations

$$K\Gamma_a^T K = -\Gamma_a, \quad K\gamma_a^T K = -\gamma_a.$$ (3.20)

Using this property of the Dirac matrices it can be easily shown that they span the orthogonal complements to the Lie algebras $\mathbf{so}(5)$ and $\mathbf{so}(4,1)$ respectively[3] Now with the choice of the coset representative given in (3.17) the current takes the form

$$J = g^{-1}dg = g^{-1}(z)g^{-1}(\theta)dg(\theta)g(z) + g^{-1}(z)dg(z).$$ (3.21)

Since

$$g(\theta)\cosh\theta + \sinh\theta, \quad g^{-1}(\theta) = \cosh\theta - \sinh\theta$$

(3.22)

we get

$$g^{-1}(\theta)dg(\theta) = (\cosh\theta - \sinh\theta)(d\cosh\theta + d\sinh\theta) = F + B,$$ (3.23)

where

$$B \equiv \cosh\theta\, d\cosh\theta - \sinh\theta\, d\sinh\theta,$$
$$F \equiv \cosh\theta\, d\sinh\theta - \sinh\theta\, d\cosh\theta$$ (3.24)

are even (contain even number of θ's) and odd (contain odd number of θ's) element respectively. With the help (3.21) and (3.23) we find that the even component of J takes the form

$$J_{\text{even}} = g^{-1}(z)Bg(z) + g^{-1}(z)dg(z)$$ (3.25)

---

[3]For very nice discussion, see [63].

while the odd element is equal to

$$J_{\text{odd}} = g^{-1}(z)Fg(z) \ . \tag{3.26}$$

As the next step we find components of the current $J^{(i)}$ that belongs to appropriate subspaces $\mathcal{H}^{(i)}$. To do this we use the relation (3.15). To present further result we define

$$G = g(z)K_8 g^t(z) = \begin{pmatrix} g_a & 0 \\ 0 & g_s \end{pmatrix} \ , \quad \tilde{G} = g(z)\tilde{K}_8 g^t(z) = \begin{pmatrix} g_a & 0 \\ 0 & -g_s \end{pmatrix} \ . \tag{3.27}$$

As was argued in [3] the 4×4 matrices $g_a \in \mathrm{SU}(2,2)$ and $g_s \in \mathrm{SU}(4)$ provide another parametrisation of the five-sphere and the AdS space. On coordinates z the global symmetry algebra is realised non-linearly. In opposite, g_a and g_s carry a linear representation of the superconformal algebra.

Now using (3.15) and (3.27) we obtain

$$2J^{(0)} = J_{\text{even}} + K_8 J_{\text{even}}^T K_8 = 2g^{-1}dg + g^{-1}(B - GB^T G^{-1} - dGG^{-1})g \tag{3.28}$$

using the fact that $K_8^{-1} = -K_8\,, \tilde{K}_8^{-1} = -\tilde{K}_8^{-1}$. In (3.28) g means $g(z)$ and in the following we use this notation. In the same way we obtain

$$2J^{(2)} = J_{\text{even}} - K_8 J_{\text{even}}^T K_8 = g^{-1}(B + GB^T G^{-1} + dGG^{-1})g \tag{3.29}$$

and

$$2J^{(1)} = J_{\text{odd}} + i\tilde{K}_8 J_{\text{odd}}^T K_8 = g^{-1}(F - i\tilde{G}F^T G^{-1})g\,,$$
$$2J^{(3)} = J_{\text{odd}} - i\tilde{K}_8 J_{\text{odd}}^T K_8 = g^{-1}(F + i\tilde{G}F^T G^{-1})g \ . \tag{3.30}$$

With the help of (3.28), (3.29) and (3.30) we can write the pure spinor Lagrangian density in the form

$$\mathcal{L} = -\frac{\sqrt{\lambda}}{8\pi}\mathrm{Str}[\frac{1}{2}\eta^{\mu\nu}(B_\mu + GB_\mu^T G^{-1} + \partial_\mu GG^{-1})(B_\nu + GB_\nu^T G^{-1} + \partial_\nu GG^{-1}) +$$
$$+\eta^{\mu\nu}(F_\mu - i\tilde{G}F_\mu^T G^{-1})(F_\nu + i\tilde{G}F_\nu^T G^{-1}) + \frac{\epsilon^{\mu\nu}}{2}(F_\mu - i\tilde{G}F_\mu^T G^{-1})(F_\nu + i\tilde{G}F_\nu^T G^{-1})]$$
$$-\frac{\sqrt{\lambda}}{2\pi}\mathrm{Str}(w_\mu \mathcal{P}^{\mu\nu}\partial_\nu\lambda + \hat{w}_\mu\tilde{\mathcal{P}}^{\mu\nu}\partial_\nu\hat{\lambda} - N_\mu\mathcal{P}^{\mu\nu}\hat{N}_\nu +$$
$$+\frac{1}{2}N_\mu\mathcal{P}^{\mu\nu}(2g^{-1}\partial_\nu g + g^{-1}(B_\nu - GB_\nu^T G^{-1} - \partial_\nu GG^{-1})g) +$$
$$+\frac{1}{2}\hat{N}_\mu\tilde{\mathcal{P}}^{\mu\nu}(2g^{-1}\partial_\nu g + g^{-1}(B_\nu - GB_\nu^T G^{-1} - \partial_\nu GG^{-1})g)) \ . \tag{3.31}$$

By using the cyclic property of the supertrace the action can be further simplified using the fact that

$$\mathrm{Str}\tilde{G}F_\mu^T G^{-1}\tilde{G}F_\nu^T G^{-1} = \mathrm{Str}F_\mu^T \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} F_\nu^T \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \mathrm{Str}F_\mu F_\nu \,, \tag{3.32}$$

where we have used

$$G^{-1}\tilde{G} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{3.33}$$

and also the fact that that F is off-diagonal matrix. Then we can simplify the action (3.31) as

$$\begin{aligned}
\mathcal{L} = &-\frac{\sqrt{\lambda}}{8\pi}\mathrm{Str}[\frac{1}{2}\eta^{\mu\nu}(B_\mu + GB_\mu^T G^{-1} + \partial_\mu GG^{-1})(B_\nu + GB_\nu^T G^{-1} + \partial_\nu GG^{-1}) + \\
&+2\eta^{\mu\nu}(F_\mu F_\nu - i\tilde{G}F_\mu^T G^{-1}F_\nu) + \epsilon^{\mu\nu}(F_\mu F_\nu - i\tilde{G}F_\mu^T G^{-1}F_\nu)] - \\
&-\frac{\sqrt{\lambda}}{2\pi}\mathrm{Str}[w_\mu \mathcal{P}^{\mu\nu}\partial_\nu\lambda + \hat{w}_\mu\tilde{\mathcal{P}}^{\mu\nu}\partial_\nu\hat{\lambda} - N_\mu\mathcal{P}^{\mu\nu}\hat{N}_\nu + \\
&+\frac{1}{2}N_\mu\mathcal{P}^{\mu\nu}(2g^{-1}\partial_\nu g + g^{-1}(B_\nu - GB_\nu^T G^{-1} - \partial_\nu GG^{-1})g) + \\
&+\frac{1}{2}\hat{N}_\mu\tilde{\mathcal{P}}^{\mu\nu}(2g^{-1}\partial_\nu g + g^{-1}(B_\nu - GB_\nu^T G^{-1} - \partial_\nu GG^{-1})g)] \,. \tag{3.34}
\end{aligned}$$

We see that the pure spinor parts of the action is rather complicated. In fact, the presence of the matrix g makes the analysis difficult since the symmetries do not act on it linearly. To resolve this problem we begin with the observation that

$$\begin{aligned}
\mathrm{Str}(N_\mu\mathcal{P}^{\mu\nu}J_\nu^{(0)}) - \mathrm{Str}(\{w_\mu,\lambda\}\mathcal{P}^{\mu\nu}J_\nu^{(0)}) &= -\mathcal{P}^{\mu\nu}\mathrm{Str}(w_\mu(\lambda J_\nu^{(0)} - J_\nu^{(0)}\lambda)) \tag{3.35} \\
&= \mathrm{Str}w_\mu\mathcal{P}^{\mu\nu}\left[J_\nu^{(0)},\lambda\right] \,,
\end{aligned}$$

where we have used the fact that the off-diagonal blocks of the matrices w, λ contain Grassman even elements. In the same we can proceed with $\hat{N}$ and then we can rewrite the pure spinor Lagrangian into the form

$$\mathcal{L}_{\mathrm{pure}} = -\frac{\sqrt{\lambda}}{2\pi}\mathrm{Str}(w_\mu\mathcal{P}^{\mu\nu}\nabla_\nu\lambda + \hat{w}_\mu\tilde{\mathcal{P}}^{\mu\nu}\nabla_\nu\hat{\lambda} - N_\mu\mathcal{P}^{\mu\nu}\hat{N}_\nu)\,, \tag{3.36}$$

where

$$\nabla_\mu X = \partial_\mu X + [J_\mu^{(0)}, X] \,. \tag{3.37}$$

The form of the current $J^{(0)}$ given in (3.28) suggests the following field redefinition of the ghost variables

$$\begin{aligned}
\lambda &= g^{-1}\overline{\lambda}g\,, & \hat{\lambda} &= g^{-1}\hat{\underline{\lambda}}g\,, \\
w_\mu &= g^{-1}\overline{w}_\mu g\,, & \hat{w}_\mu &= g^{-1}\hat{\underline{w}}_\mu g\,. \tag{3.38}
\end{aligned}$$

First of all we have to check that the new pure spinors matrices $\overline{\lambda}, \hat{\underline{\lambda}}$ still belong to $M^{(1)}, M^{(3)}$ respectively. To do this we use the fact that λ has schematically following form

$$\lambda = \begin{pmatrix} 0 & X_\lambda \\ Y_\lambda & 0 \end{pmatrix} \tag{3.39}$$

and hence

$$\overline{\lambda} = g\lambda g^{-1} = \begin{pmatrix} 0 & \tilde{g}_a^{-1}X_\lambda\tilde{g}_s \\ \tilde{g}_s^{-1}Y_\lambda\tilde{g}_a & 0 \end{pmatrix}\,, \tag{3.40}$$

where we have used the explicit form of the coset element given in (3.18). Then using the definition of Ω given in (3.9) we get

$$\Omega(\overline{\lambda}) = \begin{pmatrix} 0 & -K(\tilde{g}_s^{-1}Y_\lambda \tilde{g}_a)^T K \\ K(\tilde{g}_a^{-1}X_\lambda \tilde{g}_s)^T K & 0 \end{pmatrix}$$
$$= \begin{pmatrix} \tilde{g}_a & 0 \\ 0 & \tilde{g}_s \end{pmatrix} \begin{pmatrix} 0 & -KY_\lambda^T K \\ KX_\lambda^T K & 0 \end{pmatrix} \begin{pmatrix} \tilde{g}_a^{-1} & 0 \\ 0 & \tilde{g}_s^{-1} \end{pmatrix} = g\Omega(\lambda)g^{-1} \,. \qquad (3.41)$$

Since $\Omega(\lambda) = i\lambda$ the equation above implies

$$\Omega(\overline{\lambda}) = ig\lambda g^{-1} = i\overline{\lambda} \qquad (3.42)$$

and hence $\overline{\lambda}$ belongs to $M^{(1)}$ as well. In the same way we can show that $\underline{\hat{\lambda}}$ belongs to $M^{(3)}$ and hence the field redefinition (3.40) is well defined.

It is easy to see that if the original pure spinors $\lambda, \hat{\lambda}$ obey the pure spinor conditions then $\overline{\lambda}, \underline{\hat{\lambda}}$ obey these conditions as well. More precisely, note that the pure spinor condition for λ can be written as

$$\{\lambda, \lambda\} \, \lambda^\alpha \lambda^\beta \, \{T_\alpha, T_\beta\} \, \lambda^\alpha \lambda^\beta f_{\alpha\beta}^{\underline{c}} T_{\underline{c}} \sim \lambda^\alpha \gamma_{\alpha\beta}^{\underline{c}} \lambda^\beta T_{\underline{c}} = 0 \,. \qquad (3.43)$$

Then if we insert (3.38) into (3.43) we easily get

$$\left\{ g^{-1}\overline{\lambda}g, g^{-1}\overline{\lambda}g \right\} = g^{-1} \left\{ \overline{\lambda}, \overline{\lambda} \right\} g = 0 \Rightarrow \left\{ \overline{\lambda}, \overline{\lambda} \right\} = 0 \qquad (3.44)$$

so that $\overline{\lambda}$ obey the pure spinor constraint as well. It is clear that the same analysis can be performed for $\hat{\lambda}$ as well and we obtain that $\underline{\hat{\lambda}}$ obey the pure spinor conditions. Now with the help of (3.38) we obtain

$$w_\mu \mathcal{P}^{\mu\nu} \nabla_\nu \lambda = g^{-1}\overline{w}_\mu \mathcal{P}^{\mu\nu}(\partial_\nu \overline{\lambda} + \frac{1}{2}\left[(B - GB^T G^{-1} - dGG^{-1}), \overline{\lambda}\right])g$$
$$\hat{w}_\mu \tilde{\mathcal{P}}^{\mu\nu} \nabla_\nu \hat{\lambda} = g^{-1}\underline{\hat{w}}_\mu \tilde{\mathcal{P}}^{\mu\nu}(\partial_\nu \underline{\hat{\lambda}} + \frac{1}{2}\left[(B - GB^T G^{-1} - dGG^{-1}), \underline{\hat{\lambda}}\right])g$$
$$N_\mu = g^{-1}\overline{N}_\mu g \,,$$
$$\hat{N}_\mu = g^{-1}\underline{\hat{N}}_\mu g \,. \qquad (3.45)$$

To simplify further analysis we introduce following notation

$$J^{(0)} = g^{-1}dg + \frac{1}{2}[g^{-1}(B - GB^T G^{-1} - dGG^{-1})g]g^{-1}dg + g^{-1}\mathbf{J}^{(0)}g \,,$$
$$J^{(2)} = \frac{1}{2}[g^{-1}(B + GB^T G^{-1} + dGG^{-1})g] = g^{-1}\mathbf{J}^{(2)}g \,,$$
$$J^{(1)} = \frac{1}{2}g^{-1}(F - i\tilde{G}F^T G^{-1})g = g^{-1}\mathbf{J}^{(1)}g \,,$$
$$J^{(3)} = \frac{1}{2}g^{-1}(F + i\tilde{G}F^T G^{-1})g = g^{-1}\mathbf{J}^{(3)}g \,. \qquad (3.46)$$

The we can write the pure spinor action in the same form as in (3.1)

$$S = -\frac{\sqrt{\lambda}}{2\pi} \int d\tau d\sigma \sqrt{-\eta} \text{Str}[\frac{1}{2}\eta^{\mu\nu} \left(\mathbf{J}_\mu^{(2)}\mathbf{J}_\nu^{(2)} + \mathbf{J}_\mu^{(1)}\mathbf{J}_\nu^{(3)} + \mathbf{J}_\mu^{(3)}\mathbf{J}_\nu^{(1)} \right) +$$

$$+ \frac{\epsilon^{\mu\nu}}{4} \left(\mathbf{J}_\mu^{(1)}\mathbf{J}_\nu^{(3)} - \mathbf{J}_\mu^{(3)}\mathbf{J}_\nu^{(1)} \right)] + S_{\text{ghost}} \,,$$

$$S_{\text{ghost}} = -\frac{\sqrt{\lambda}}{2\pi} \int d\tau d\sigma \sqrt{-\eta} \text{Str}[\overline{w}_\mu \mathcal{P}^{\mu\nu}\partial_\nu \overline{\lambda} + \underline{\hat{w}}_\mu \tilde{\mathcal{P}}^{\mu\nu}\partial_\nu \underline{\hat{\lambda}} +$$

$$+ \overline{N}_\mu \mathcal{P}^{\mu\nu}\mathbf{J}_\nu^{(0)} + \underline{\hat{N}}_\mu \tilde{\mathcal{P}}^{\mu\nu}\mathbf{J}_\nu^{(0)} - \frac{1}{2}\overline{N}_\mu \mathcal{P}^{\mu\nu}\underline{\hat{N}}_\nu - \frac{1}{2}\underline{\hat{N}}_\mu \tilde{\mathcal{P}}^{\mu\nu}\overline{N}_\nu] \,. \tag{3.47}$$

However there is one crucial difference between the action (3.47) and (3.1). Due to the explicit form of the coset representative (3.17) it is clear that the currents $\mathbf{J}^{(i)}$ do not transform under the gauge transformations as the original one $J^{(i)}$. More precisely, the original action (3.1) was invariant under the gauge transformations

$$J' = h^{-1}Jh + h^{-1}dh \,, \quad \lambda' = h^{-1}\lambda h \,, \quad \hat{\lambda}' = h^{-1}\hat{\lambda}h \,,$$

$$w'_\mu = h^{-1}w_\mu h \,, \quad \hat{w}'_\mu = h^{-1}\hat{w}_\mu h \,, \tag{3.48}$$

where h belongs to SO(4, 1) × SO(5). Clearly the redefined currents $\mathbf{J}$ and ghost variables do not transform in the same way as (3.48). This follows from the fact that the choice of given coset representative effectively fixes given gauge symmetry. For that reason the action (3.47) does not possess the gauge symmetry of the original action.

We conclude this section with the brief discussion of the properties of the matrix G. With the certain choice of the matrix K the matrix g_s parameterising S^5 can be written as follows

$$g_s = \begin{pmatrix} 0 & u_3 & u_1 & u_2 \\ -u_3 & 0 & u_2^* & -u_1^* \\ -u_1 & -u_2^* & 0 & u_3^* \\ -u_2 & u_1^* & -u_3^* & 0 \end{pmatrix} \,. \tag{3.49}$$

This is unitary matrix $g_s^\dagger g_s = 1$ on the condition that the three complex coordinates u_i obey the constraint $|u_1|^2 + |u_2|^2 + |u_3|^2 = 1$. A similar parameterisation of the AdS_5 space is given by

$$g_a = \begin{pmatrix} 0 & v_3 & v_1 & v_2 \\ -v_3 & 0 & -v_2^* & v_1^* \\ -v_1 & v_2^* & 0 & v_3^* \\ -v_2 & -v_1^* & -v_3^* & 0 \end{pmatrix} \,, \tag{3.50}$$

where now $g_a \in$ SU(2, 2) so that it obeys $g_a^\dagger E g_a = E$ where $E = \text{diag}(1, 1, -1, -1)$ provided the complex numbers v_i satisfy the constraint $|v_1|^2 + |v_2|^2 - |v_3|^2 = -1$.

4. Equation of motions and BRST invariance

Our goal is to express the equations of motion that follow from the action (3.1) in terms of redefined ghost fields (3.38) and of the currents $\mathbf{J}^{(i)}$ defined in (3.46). We firstly write

the equations of motion that arise from (3.1). These equations of motion were determined previously in [47] and their covariant formulation was also given in [55]

$$\tilde{\mathcal{P}}^{\mu\nu}\nabla_\mu J_\nu^{(3)} + [J_\nu^{(3)}, N_\mu]\mathcal{P}^{\mu\nu} + [J_\nu^{(3)}, \hat{N}_\mu]\tilde{\mathcal{P}}^{\mu\nu} = 0\,,$$
$$\mathcal{P}^{\mu\nu}\nabla_\mu J_\nu^{(1)} + [J_\nu^{(1)}, N_\mu]\mathcal{P}^{\mu\nu} + [J_\nu^{(1)}, \hat{N}_\mu]\tilde{\mathcal{P}}^{\mu\nu} = 0\,,$$
$$\mathcal{P}^{\mu\nu}\nabla_\mu J_\nu^{(2)} - \epsilon^{\mu\nu}[J_\mu^{(1)}, J_\nu^{(1)}] + [J_\nu^{(2)}, N_\mu]\mathcal{P}^{\mu\nu} + [J_\nu^{(2)}, \hat{N}_\mu]\tilde{\mathcal{P}}^{\mu\nu} = 0\,,$$
$$\tilde{\mathcal{P}}^{\mu\nu}\nabla_\mu J_\nu^{(2)} + \epsilon^{\mu\nu}[J_\mu^{(3)}, J_\nu^{(3)}] + [J_\nu^{(2)}, N_\mu]\mathcal{P}^{\mu\nu} + [J_\nu^{(2)}, \hat{N}_\mu]\tilde{\mathcal{P}}^{\mu\nu} = 0\,,$$
$$\mathcal{P}^{\mu\nu}\nabla_\nu\lambda + \mathcal{P}^{\mu\nu}[\lambda, \hat{N}_\nu] = 0\,,$$
$$\tilde{\mathcal{P}}^{\mu\nu}\nabla_\nu\hat{\lambda} + \tilde{\mathcal{P}}^{\mu\nu}[\hat{\lambda}, N_\nu] = 0\,, \tag{4.1}$$

where

$$\nabla_\nu J_\mu^{(i)} = \partial_\nu J_\mu^{(i)} + [J_\nu^{(0)}, J_\mu^{(i)}]\,,$$
$$\nabla_\mu\lambda = \partial_\mu\lambda + \left[J_\mu^{(0)}, \lambda\right]\,. \tag{4.2}$$

Now we rewrite these equations of motion using the form of the currents given in (3.46) and we obtain

$$\nabla_\mu J_\nu^{(i)} = g^{-1}(\partial_\mu \mathbf{J}_\nu^{(i)} + \left[\mathbf{J}_\mu^{(0)}, J_\nu^{(i)}\right])g \equiv g^{-1}\overline{\nabla}_\nu J_\mu^{(i)} g\,, \quad i = 1, 2, 3\,,$$
$$\nabla_\mu\lambda = g^{-1}\overline{\nabla}_\mu\overline{\lambda}g\,, \quad \nabla_\mu\hat{\lambda} = g^{-1}\overline{\nabla}_\mu\underline{\hat{\lambda}}g\,. \tag{4.3}$$

Then it is easy to see that the equations of motion given above take the form

$$\tilde{\mathcal{P}}^{\mu\nu}\overline{\nabla}_\mu\mathbf{J}_\nu^{(3)} + [\mathbf{J}_\nu^{(3)}, \overline{N}_\mu]\mathcal{P}^{\mu\nu} + [\mathbf{J}_\nu^{(3)}, \underline{\hat{N}}_\mu]\tilde{\mathcal{P}}^{\mu\nu} = 0\,, \tag{4.4}$$
$$\mathcal{P}^{\mu\nu}\overline{\nabla}_\mu\mathbf{J}_\nu^{(1)} + [\mathbf{J}_\nu^{(1)}, \overline{N}_\mu]\mathcal{P}^{\mu\nu} + [\mathbf{J}_\nu^{(1)}, \underline{\hat{N}}_\mu]\tilde{\mathcal{P}}^{\mu\nu} = 0\,, \tag{4.5}$$
$$\mathcal{P}^{\mu\nu}\overline{\nabla}_\mu\mathbf{J}_\nu^{(2)} - \epsilon^{\mu\nu}[\mathbf{J}_\mu^{(1)}, \mathbf{J}_\nu^{(1)}] + [\mathbf{J}_\nu^{(2)}, \overline{N}_\mu]\mathcal{P}^{\mu\nu} + [\mathbf{J}_\nu^{(2)}, \underline{\hat{N}}_\mu]\tilde{\mathcal{P}}^{\mu\nu} = 0\,, \tag{4.6}$$
$$\tilde{\mathcal{P}}^{\mu\nu}\overline{\nabla}_\mu\mathbf{J}_\nu^{(2)} + \epsilon^{\mu\nu}[\mathbf{J}_\mu^{(3)}, \mathbf{J}_\nu^{(3)}] + [\mathbf{J}_\nu^{(2)}, \overline{N}_\mu]\mathcal{P}^{\mu\nu} + [\mathbf{J}_\nu^{(2)}, \underline{\hat{N}}_\mu]\tilde{\mathcal{P}}^{\mu\nu} = 0\,, \tag{4.7}$$
$$\mathcal{P}^{\mu\nu}\overline{\nabla}_\nu\overline{\lambda} + \mathcal{P}^{\mu\nu}[\overline{\lambda}, \underline{\hat{N}}_\nu] = 0\,, \tag{4.8}$$
$$\tilde{\mathcal{P}}^{\mu\nu}\overline{\nabla}_\nu\underline{\hat{\lambda}} + \tilde{\mathcal{P}}^{\mu\nu}[\underline{\hat{\lambda}}, \overline{N}_\nu] = 0\,. \tag{4.9}$$

The fact that in the new variables the equations of motion have the same form as the equations given in (4.1) has an important consequence for the conservation of the BRST currents. These currents are defined as [47]

$$j_R^\mu = \mathrm{Str}(\hat{\lambda}J_\nu^{(1)}\tilde{\mathcal{P}}^{\nu\mu})\,, \quad j_L^\mu = \mathrm{Str}(\lambda J_\nu^{(3)}\mathcal{P}^{\nu\mu}) \tag{4.10}$$

and they are conserved

$$\partial_\mu j_{R,L}^\mu = 0\,. \tag{4.11}$$

With the help of (3.38) and (3.46) we can rewrite (4.10) into the form

$$j_R^\mu = \mathrm{Str}(\underline{\hat{\lambda}}\mathbf{J}_\nu^{(1)}\tilde{\mathcal{P}}^{\nu\mu})\,, \quad j_L^\mu = \mathrm{Str}(\overline{\lambda}\mathbf{J}_\nu^{(3)}\mathcal{P}^{\nu\mu})\,. \tag{4.12}$$

Now we will show that these currents are conserved as well. To do this we will calculate $\partial_\mu j_L^\mu$ using the equations of motion (4.4) and (4.8)

$$
\begin{aligned}
\partial_\mu j_L^\mu &= \mathrm{Str}(\mathcal{P}^{\nu\mu}\partial_\mu\overline{\lambda}\mathbf{J}_\nu^{(3)}) + \mathrm{Str}(\overline{\lambda}\tilde{\mathcal{P}}^{\mu\nu}\partial_\mu\mathbf{J}_\nu^{(3)}) \\
&= \mathrm{Str}((-\mathcal{P}^{\nu\mu}[\hat{\underline{\lambda}},\hat{\underline{N}}_\mu] - \mathcal{P}^{\mu\nu}[\mathbf{J}_\mu^{(0)},\overline{\lambda}])\mathbf{J}_\nu^{(3)}) + \\
&\quad + \mathrm{Str}(\overline{\lambda}(-\tilde{\mathcal{P}}^{\mu\nu}[\mathbf{J}_\mu^{(0)},\mathbf{J}_\nu^{(3)}] - [\mathbf{J}_\nu^{(3)},\overline{N}_\mu]\mathcal{P}^{\mu\nu} - [\mathbf{J}_\nu^{(3)},\hat{N}_\mu]\tilde{\mathcal{P}}^{\mu\nu}) \\
&= -\mathrm{Str}(\mathbf{J}_\nu^{(3)}\left[\overline{N}_\mu,\overline{\lambda}\right])\mathcal{P}^{\mu\nu}
\end{aligned}
\tag{4.13}
$$

that vanishes thanks to the pure spinor constraint. In the same way we can prove the conservation of j_R^μ. The existence of two conserved BRST currents (4.12) imply that we can define two BRST charges

$$
Q_L = \frac{1}{2\pi}\int_0^{2\pi} d\sigma j_L^0, \quad Q_R = \frac{1}{2\pi}\int_0^{2\pi} d\sigma j_R^0 \; .
\tag{4.14}
$$

Let us now calculate their time derivative

$$
\begin{aligned}
\frac{dQ_L}{d\tau} &= -\frac{1}{2\pi}\int_0^{2\pi} d\sigma\partial_\sigma j_L^1 = -\frac{1}{2\pi}(j_L^1(2\pi) - j_L^1(0))\,, \\
\frac{dQ_R}{d\tau} &= -\frac{1}{2\pi}\int_0^{2\pi} d\sigma\partial_\sigma j_R^1 = -\frac{1}{2\pi}(j_R^1(2\pi) - j_R^1(0))\,,
\end{aligned}
\tag{4.15}
$$

where we have used (4.11). For ordinary closed string we demand that the world-sheet fields are periodic

$$
j_{L,R}^1(\tau,\sigma + 2\pi) = j_{L,R}^1(\tau,\sigma) \; .
\tag{4.16}
$$

Then (4.15) implies that Q_L, Q_R are time-independent. Even if these results are well known we reviewed here since as we will see in the next section the world-volume modes do not have to be periodic and hence the existence of the conserved BRST charges is not generally obvious.

5. Fermions and pure spinors twisting

The original fermions and pure spinors transform under isometries of the five sphere. To apply the approach presented in section (2) we need to redefine the fermions and pure spinor in such a way that they become neutral under the isometries. The twisted boundary conditions for the original angles of $AdS_5 \times S^5$ will induce twisted boundary conditions for the original charged fermions and pure spinors of $AdS_5 \times S^5$.

To proceed we will closely follow [61] since it turns out that the approach presented here can be easily extended to the pure spinor string as well. We begin with the study of the invariance of the Lagrangian under the abelian subalgebra of the superconformal group. The bosonic symmetry algebra $SO(4,2) \times SO(6)$ has six Cartan generators: three for $SO(4,2)$ and three for $SO(6)$. If we introduce the polar representation

$$
u_i = r_i e^{i\phi_i}, \quad v_i = \rho_i e^{i\psi_i} \; ,
\tag{5.1}
$$

where r_i, ρ_i are real, then the six commuting isometries are realised as constant shift of the angle variables

$$\phi_i' = \phi_i + \epsilon_i \,, \quad \psi_i' = \psi_i + \tilde{\epsilon}_i \,. \tag{5.2}$$

It is remarkable that matrices g_s and g_a enjoy the following factorisation property [2, 65, 66]

$$\begin{aligned} g_s(r, \phi) &= M(\phi)\hat{g}_s(r)M(\phi) \,, \\ g_a(r, \psi) &= M(\psi)\hat{g}_a(\rho)M(\psi) \,, \end{aligned} \tag{5.3}$$

where

$$\hat{g}_s(r) = \begin{pmatrix} 0 & r_3 & r_1 & r_2 \\ -r_3 & 0 & r_2 & -r_1 \\ -r_1 & -r_2 & 0 & -r_3 \\ -r_2 & r_1 & r_3 & 0 \end{pmatrix}, \quad \hat{g}_a(r) = \begin{pmatrix} 0 & \rho_3 & \rho_1 & \rho_2 \\ -\rho_3 & 0 & \rho_2 & -\rho_1 \\ -\rho_1 & -\rho_2 & 0 & \rho_3 \\ -\rho_2 & \rho_1 & -\rho_3 & 0 \end{pmatrix}, \tag{5.4}$$

and where $M(\phi) = e^{\frac{i}{2}\Phi}$ with $\Phi = \mathrm{diag}(\Phi_1, \ldots, \Phi_4)$ where Φ_i are equal to

$$\begin{aligned} \Phi_1 &= \phi_1 + \phi_2 + \phi_3 \,, \\ \Phi_2 &= -\phi_1 - \phi_2 + \phi_3 \,, \\ \Phi_3 &= \phi_1 - \phi_2 - \phi_3 \,, \\ \Phi_4 &= -\phi_1 + \phi_2 - \phi_3 \,. \end{aligned} \tag{5.5}$$

Note that in this case the matrix $G, \tilde{G}$ can be written as

$$G = M\hat{G}M \,, \quad \hat{G} = \begin{pmatrix} \hat{g}_a & 0 \\ 0 & \hat{g}_s \end{pmatrix}, \tag{5.6}$$

$$\tilde{G} = M\hat{\tilde{G}}M \,, \quad \hat{\tilde{G}} = \begin{pmatrix} \hat{g}_a & 0 \\ 0 & -\hat{g}_s \end{pmatrix}, \tag{5.7}$$

where

$$M = \begin{pmatrix} M(\psi) & 0 \\ 0 & M(\phi) \end{pmatrix}. \tag{5.8}$$

If we insert (5.6) and (5.7) to the action (3.47) we obtain that the action explicitly depends on Φ. This fact precludes to perform the analysis given in section (2). In order to obtain the sigma model when the fermions and pure spinors are spectators we have to perform their redefinition.

In order to find the fermionic and pure spinor redefinition note that fermions and pure spinor matrices can be written as

$$\theta = \begin{pmatrix} 0 & X_\theta \\ Y_\theta & 0 \end{pmatrix}, \quad \lambda = \begin{pmatrix} 0 & X_\lambda \\ Y_\lambda & 0 \end{pmatrix}, \quad \hat{\lambda} = \begin{pmatrix} 0 & X_{\hat{\lambda}} \\ Y_{\hat{\lambda}} & 0 \end{pmatrix}, \tag{5.9}$$

where in the case of $\lambda, \hat{\lambda}$ the off-diagonal matrices $X_\lambda, Y_\lambda, X_{\hat{\lambda}}, Y_{\hat{\lambda}}$ are bosonic. We must however stress that $\lambda, \hat{\lambda}$ are not odd matrices of $\mathbf{su}(2,2|4)$ superalgebra. This follows from the fact that they are defined as $\lambda = \lambda^\alpha T_\alpha$ where crucially λ^α is complex number while for an element from $\mathbf{su}(2,2|4)$ this parameter should be real. In fact it can be easily seen that if λ^α were real the solution of the pure spinor constraint would be trivial.

In case of fermions we perform following rescaling

$$X_\theta = M(\psi)\hat{X}M(\phi)^{-1}, \quad Y_\theta = M(\phi)\hat{X}M(\psi)^{-1}. \tag{5.10}$$

Then it follows that

$$g(\theta) = M\hat{g}(\hat{\theta})M^{-1}, \tag{5.11}$$

where we have defined

$$M \equiv \begin{pmatrix} M(\psi) & 0 \\ 0 & M(\phi) \end{pmatrix}, \tag{5.12}$$

and where the fermions $\hat{\theta}$ are uncharged under all $\mathrm{U}(1)$s.

Using this redefinition the currents (3.46) take the form

$$\mathbf{J}^{(0)} = \frac{1}{2}M(\hat{B} - \hat{G}\hat{B}^T\hat{G}^{-1} - d\hat{G}\hat{G}^{-1} - \frac{i}{2}d\Phi - \frac{i}{2}\hat{G}d\Phi\hat{G}^{-1})M^{-1} \equiv M\tilde{\mathbf{J}}^{(0)}M^{-1},$$

$$\mathbf{J}^{(2)} = \frac{1}{2}M(\hat{B} + \hat{G}\hat{B}^T\hat{G}^{-1} + \frac{i}{2}d\Phi + \frac{i}{2}\hat{G}d\Phi\hat{G}^{-1})M^{-1} \equiv M\tilde{\mathbf{J}}^{(2)}M^{-1},$$

$$\mathbf{J}^{(1)} = \frac{1}{2}M[\hat{F} - \hat{\tilde{G}}\hat{F}^T\hat{G}]M^{-1} \equiv M\tilde{\mathbf{J}}^{(1)}M^{-1},$$

$$\mathbf{J}^{(3)} = M[\hat{F} + i\hat{\tilde{G}}\hat{F}^T\hat{G}^{-1}]M^{-1} \equiv M\tilde{\mathbf{J}}^{(3)}M^{-1} \tag{5.13}$$

and consequently the matter part of the pure spinor action takes the form

$$S = -\frac{\sqrt{\lambda}}{2\pi}\int d\tau d\sigma\sqrt{-\eta}\,\mathrm{Str}[\frac{1}{2}\eta^{\mu\nu}\left(\tilde{\mathbf{J}}_\mu^{(2)}\tilde{\mathbf{J}}_\nu^{(2)} + \tilde{\mathbf{J}}_\mu^{(1)}\tilde{\mathbf{J}}_\nu^{(3)} + \tilde{\mathbf{J}}_\mu^{(3)}\tilde{\mathbf{J}}_\nu^{(1)}\right) +$$
$$+ \frac{\epsilon^{\mu\nu}}{4}\left(\tilde{\mathbf{J}}_\mu^{(1)}\tilde{\mathbf{J}}_\nu^{(3)} - \tilde{\mathbf{J}}_\mu^{(3)}\tilde{\mathbf{J}}_\nu^{(1)}\right)]. \tag{5.14}$$

It is important that the action (5.14) depends on Φ through the expression of $d\Phi$ only and hence it is invariant under the shift $\Phi' = \Phi + \epsilon$. In other words matter part of the pure spinor string in the $AdS_5 \times S^5$ background takes the form of the sigma model action studied in the section (2) and consequently the TsT transformation can be performed.

In the similar way as in case of fermions we propose the following redefinition of the ghost variables

$$\overline{\lambda} = M\tilde{\lambda}M^{-1}, \quad \underline{\hat{\lambda}} = M\tilde{\hat{\lambda}}M^{-1} \tag{5.15}$$

and

$$\tilde{w}_\mu = M\overline{w}_\mu M^{-1}, \quad \tilde{\hat{w}}_\mu = M\underline{\hat{w}}_\mu M^{-1}, \tag{5.16}$$

where $\tilde{\lambda}, \tilde{\hat{\lambda}}, \tilde{w}_\mu, \tilde{\hat{w}}_\mu$ are not charged under $\mathrm{U}(1)$'s isometries. Note that (5.15) and (5.16) imply that $\tilde{N}_\mu, \tilde{\hat{N}}_\mu$ are neutral under $\mathrm{U}(1)'s$ isometries as well. Further, if we insert (5.15) into the pure spinor constraint we obtain

$$\left\{M\tilde{\lambda}M^{-1}, M\tilde{\lambda}M^{-1}\right\}M\left\{\tilde{\lambda}, \tilde{\lambda}\right\}M^{-1} = 0 \tag{5.17}$$

and hence $\tilde{\lambda}$ obeys the pure spinor constraints. It is also clear that this analysis holds for $\hat{\tilde{\lambda}}$ as well. Finally we determine the form of $\overline{\nabla}_\mu \overline{\lambda}$

$$\overline{\nabla}_\mu \overline{\lambda} = \partial_\mu \overline{\lambda} + \left[\mathbf{J}_\mu^{(0)}, \overline{\lambda} \right]$$

$$= M(\partial_\mu \tilde{\lambda} + \frac{i}{2}[\partial_\mu \Phi, \tilde{\lambda}] + [\tilde{\mathbf{J}}_\mu^{(0)}, \tilde{\lambda}])M^{-1} \equiv M\tilde{\nabla}_\mu \tilde{\lambda} M^{-1} \tag{5.18}$$

using

$$d\overline{\lambda} = M \left(d\tilde{\lambda} + \frac{i}{2} d\Phi \tilde{\lambda} - \frac{i}{2} \tilde{\lambda} d\Phi \right) M^{-1} . \tag{5.19}$$

Clearly the same equation holds for $\hat{\underline{\lambda}}, \hat{\underline{w}}_\mu$. In summary we obtain following form of the pure spinor Lagrangian from (3.47)

$$\mathcal{L}_{\text{pure}} = -\frac{\sqrt{\lambda}}{2} \text{Str}[\tilde{w}_\mu \mathcal{P}^{\mu\nu} \tilde{\nabla}_\nu \tilde{\lambda} + \hat{\tilde{w}}_\mu \tilde{\mathcal{P}}^{\mu\nu} \tilde{\nabla}_\nu \hat{\tilde{\lambda}} - \tilde{N}_\mu \tilde{\mathcal{P}}^{\mu\nu} \hat{\tilde{N}}_\nu] . \tag{5.20}$$

We see that (5.20) depends on Φ through $d\Phi$ only and hence the analysis performed in section (2) can be applied for pure spinor action as well.

It will be useful to express the equations of motion for $\mathbf{J}$ and ghosts $\overline{\lambda}, \hat{\underline{\lambda}}, \overline{w}, \hat{\underline{w}}$ that were given in (4.4), (4.5), (4.6), (4.7), (4.8) and (4.9) in terms of the variables defined in (5.13), (5.15) and (5.16). As the first step we express the covariant derivative $\overline{\nabla}\mathbf{J}^{(i)}$ using the redefined currents (5.13)

$$\overline{\nabla}\mathbf{J}^{(i)} = M(d\tilde{\mathbf{J}}^{(i)} + \frac{i}{2} \left[d\Phi, \tilde{\mathbf{J}}^{(i)} \right] + \left[\tilde{\mathbf{J}}^{(0)}, \tilde{\mathbf{J}}^{(i)} \right])M^{-1} \equiv M\tilde{\nabla}\tilde{\mathbf{J}}^{(i)}M^{-1} , \tag{5.21}$$

where we have introduced the derivative $\tilde{\nabla}$ that by definition depends on $\tilde{\mathbf{J}}^{(0)}$ and on the derivative of Φ. Then with the help of (5.13), (5.15), (5.16) and (5.21) we can determine from (4.4), (4.5), (4.6), (4.7), (4.8) and (4.9) the equations of motion for $\tilde{\mathbf{J}}, \tilde{\lambda}, \hat{\tilde{\lambda}}, \tilde{w}_\mu$ and $\hat{\tilde{w}}_\mu$ in the form

$$\tilde{\mathcal{P}}^{\mu\nu} \tilde{\nabla}_\mu \tilde{\mathbf{J}}_\nu^{(3)} + [\tilde{\mathbf{J}}_\nu^{(3)}, \tilde{N}_\mu]\mathcal{P}^{\mu\nu} + [\tilde{\mathbf{J}}_\nu^{(3)}, \hat{\tilde{N}}_\mu]\tilde{\mathcal{P}}^{\mu\nu} = 0 ,$$

$$\mathcal{P}^{\mu\nu} \tilde{\nabla}_\mu \tilde{\mathbf{J}}_\nu^{(1)} + [\tilde{\mathbf{J}}_\nu^{(1)}, \tilde{N}_\mu]\mathcal{P}^{\mu\nu} + [\tilde{\mathbf{J}}_\nu^{(1)}, \hat{\tilde{N}}_\mu]\tilde{\mathcal{P}}^{\mu\nu} = 0 ,$$

$$\mathcal{P}^{\mu\nu} \tilde{\nabla}_\mu \tilde{\mathbf{J}}_\nu^{(2)} - \epsilon^{\mu\nu}[\tilde{\mathbf{J}}_\mu^{(1)}, \tilde{\mathbf{J}}_\nu^{(1)}] + [\tilde{\mathbf{J}}_\nu^{(2)}, \tilde{N}_\mu]\mathcal{P}^{\mu\nu} + [\tilde{\mathbf{J}}_\nu^{(2)}, \hat{\tilde{N}}_\mu]\tilde{\mathcal{P}}^{\mu\nu} = 0 ,$$

$$\tilde{\mathcal{P}}^{\mu\nu} \tilde{\nabla}_\mu \tilde{\mathbf{J}}_\nu^{(2)} + \epsilon^{\mu\nu}[\tilde{\mathbf{J}}_\mu^{(3)}, \tilde{\mathbf{J}}_\nu^{(3)}] + [\tilde{\mathbf{J}}_\nu^{(2)}, \tilde{N}_\mu]\mathcal{P}^{\mu\nu} + [\tilde{\mathbf{J}}_\nu^{(2)}, \hat{\tilde{N}}_\mu]\tilde{\mathcal{P}}^{\mu\nu} = 0 ,$$

$$\mathcal{P}^{\mu\nu} \tilde{\nabla}_\nu \tilde{\lambda} + \mathcal{P}^{\mu\nu}[\tilde{\lambda}, \hat{\tilde{N}}_\nu] = 0 ,$$

$$\tilde{\mathcal{P}}^{\mu\nu} \tilde{\nabla}_\nu \hat{\tilde{\lambda}} + \tilde{\mathcal{P}}^{\mu\nu}[\hat{\tilde{\lambda}}, \tilde{N}_\nu] = 0 . \tag{5.22}$$

Finally we will discuss the conservation of the BRST currents given in (4.12). With the help of (5.13) and (5.15) it is easy to see that they are equal to

$$\tilde{j}_R^\mu = \text{Str}(\hat{\tilde{\lambda}} \tilde{\mathbf{J}}_\nu^{(1)} \tilde{\mathcal{P}}^{\nu\mu}) , \quad \tilde{j}_L^\mu = \text{Str}(\tilde{\lambda} \tilde{\mathbf{J}}_\nu^{(3)} \mathcal{P}^{\nu\mu}) \tag{5.23}$$

and that they are again conserved as a consequence of the equations of motion (5.22). Consequently the time derivative of the BRST charges is equal to

$$\frac{dQ_L}{d\tau} = -\frac{1}{2\pi}(\tilde{j}_L^1(2\pi) - \tilde{j}_L^1(0)) , \quad \frac{dQ_R}{d\tau} = -\frac{1}{2\pi}(\tilde{j}_R^1(2\pi) - \tilde{j}_R^1(0)) . \tag{5.24}$$

6. TsT transformation on the five sphere

Even if the general analysis performed above is valid for the TsT transformation in the AdS_5 space as well we restrict to the TsT transformation applied to the five-sphere, following [61]. This restriction implies that we do need to impose that fermions and pure spinors are neutral under isometries of AdS_5. Then we can take $M(\psi) = 1$ and hence we obtain that the matrix M takes the form

$$M = \begin{pmatrix} 1 & 0 \\ 0 & M(\phi) \end{pmatrix} . \tag{6.1}$$

In order to determine the twisted boundary conditions for fermions and pure spinors we have to take into account that the redefined fermions and the pure spinors do not transform under the TsT transformations. Therefore the original charged fermions in $AdS_5 \times S^5$ and pure spinors satisfy the twisted boundary conditions. We find these boundary conditions using the relation between $\hat{\theta}$ and θ and the twisted boundary condition for angle ϕ_i that has impact on the matrix M since

$$\phi_i(2\pi) = \phi_i(0) + 2\pi(n_i - \nu_i), \quad \nu_i = \epsilon_{ijk}\gamma_j J_k \tag{6.2}$$

or equivalently

$$\begin{aligned} \Phi_1(2\pi) &= \Phi_1(0) + 2\pi(n_1 + n_2 + n_3 - \nu_1 - \nu_2 - \nu_3) \equiv \Phi_1(0) - 2\pi\Lambda_1 \,, \\ \Phi_2(2\pi) &= \Phi_2(0) + 2\pi(-n_1 - n_2 + n_3 + \nu_1 + \nu_2 - \nu_3) \equiv \Phi_2(0) - 2\pi\Lambda_2 \,, \\ \Phi_3(2\pi) &= \Phi_3(0) + 2\pi(n_1 - n_2 - n_3 - \nu_1 + \nu_2 + \nu_3) \equiv \Phi_3(0) - 2\pi\Lambda_3 \,, \\ \Phi_4(2\pi) &= \Phi_4(0) + 2\pi(-n_1 + n_2 - n_3 + \nu_1 - \nu_2 + \nu_3) \equiv \Phi_4(0) - 2\pi\Lambda_4 \,. \end{aligned} \tag{6.3}$$

Using (6.3) we easily obtain

$$M(\Phi(2\pi)) = \begin{pmatrix} e^{-i\pi\Lambda_1} & 0 & 0 & 0 \\ 0 & e^{-i\pi\Lambda_2} & 0 & 0 \\ 0 & 0 & e^{-i\pi\Lambda_3} & 0 \\ 0 & 0 & 0 & e^{-i\pi\Lambda_4} \end{pmatrix} \begin{pmatrix} e^{i\Phi_1(0)} & 0 & 0 & 0 \\ 0 & e^{i\Phi_2(0)} & 0 & 0 \\ 0 & 0 & e^{i\Phi_3(0)} & 0 \\ 0 & 0 & 0 & e^{i\Phi_4(0)} \end{pmatrix} \tag{6.4}$$

or in compact notation

$$M(\Phi(2\pi)) = e^{-i\pi\Lambda}M(\Phi(0)), \tag{6.5}$$

where $\Lambda = \mathrm{diag}(\Lambda_1, \Lambda_2, \Lambda_3, \Lambda_4)$. Then we have

$$\begin{aligned} g(\theta)(2\pi) &= M(2\pi)g(\hat{\theta})(2\pi)M^{-1}(2\pi) \\ &= \begin{pmatrix} 1 & 0 \\ 0 & e^{-i\pi\Lambda} \end{pmatrix} M(\Phi(0))g(\hat{\theta}(0))M^{-1}(\Phi(0)) \begin{pmatrix} 1 & 0 \\ 0 & e^{i\pi\Lambda} \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & e^{-i\pi\Lambda} \end{pmatrix} g(\theta)(0) \begin{pmatrix} 1 & 0 \\ 0 & e^{i\pi\Lambda} \end{pmatrix} \end{aligned} \tag{6.6}$$

using the fact that $\hat{\theta}$ do not transform under TsT duality and hence they are the same in TsT dual background with standard periodicity $\hat{\theta}(2\pi) = \hat{\theta}(0)$.

Now we would like to explain carefully our calculations. [4] We have derived the pure spinor action in the $AdS_5 \times S^5$ given in (5.14) and (5.20) that by construction is manifestly invariant under the isometry of the background parametrised by Φ. Now let us suppose that we have pure spinor action that describes *closed string* in the γ-deformed background. Since the γ-deformed background can be derived from the original $AdS_5 \times S^5$ by sequence of the TsT transformations the analysis performed in section (2) suggests that this action has the same functional form as the action given in (5.14) and (5.20). Let us denote the corresponding Lagrangian as $\mathcal{L}(\tilde{\mathbf{J}}^0, \tilde{\lambda}, \hat{\tilde{\lambda}})$ where superscripts on $\tilde{\mathbf{J}}$ mean that these currents explicitly depend on the γ-deformed background. According to the arguments given in the section (2) this Lagrangian can be mapped by sequence of TsT transformations to the Lagrangian $\mathcal{L}(\tilde{\mathbf{J}}, \tilde{\lambda}, \hat{\tilde{\lambda}})$ where now the angle variables obey twisted boundary conditions according to (6.5). On the other hand the fermionic $\hat{\theta}$ and ghost variables $\tilde{\lambda}, \hat{\tilde{\lambda}}, \tilde{w}, \hat{\tilde{w}}$ have periodic boundary conditions since they are neutral under $\mathrm{U}(1)'s$ isometries. Then the form of the currents $\tilde{\mathbf{J}}$ given in (5.13) imply that they are periodic since they depend on r, ρ and $\hat{\theta}$ and as we argued above these modes are periodic. It is also easy to see that $d\Phi(2\pi) = d\Phi(0)$. Explicitly, we have

$$
\begin{aligned}
\tilde{\mathbf{J}}^{(i)}(2\pi) &= \tilde{\mathbf{J}}^{(i)}(0)\,, & i &= 0, 1, 2, 3\,, \\
\tilde{\lambda}(2\pi) &= \tilde{\lambda}(0)\,, & \hat{\tilde{\lambda}}(2\pi) &= \hat{\tilde{\lambda}}(0)\,, \\
\tilde{w}_\mu(2\pi) &= \tilde{w}_\mu(0)\,, & \hat{\tilde{w}}_\mu(2\pi) &= \hat{\tilde{w}}_\mu(0)\,.
\end{aligned}
\tag{6.7}
$$

These boundary conditions immediately show that the conserved BRST currents given in (5.23) imply the existence of the time-independent BRST charges as follows from (5.24).

On the other hand we can take one step further and study the pure spinor action expressed with the variables $\mathbf{J}, \overline{\lambda}, \hat{\underline{\lambda}}$. These variables now obey twisted boundary conditions as follows from (5.13), (5.15) and (5.16)

$$
\begin{aligned}
\mathbf{J}^{(i)}(2\pi) &= \begin{pmatrix} 1 & 0 \\ 0 & e^{-i\pi\Lambda} \end{pmatrix} \mathbf{J}^{(i)}(0) \begin{pmatrix} 1 & 0 \\ 0 & e^{i\pi\Lambda} \end{pmatrix}\,, & i &= 0, 1, 2, 3 \\
\overline{\lambda}(2\pi) &= \begin{pmatrix} 1 & 0 \\ 0 & e^{-i\pi\Lambda} \end{pmatrix} \overline{\lambda}(0) \begin{pmatrix} 1 & 0 \\ 0 & e^{i\pi\Lambda} \end{pmatrix}\,, & \hat{\underline{\lambda}}(2\pi) &= \begin{pmatrix} 1 & 0 \\ 0 & e^{-i\pi\Lambda} \end{pmatrix} \hat{\underline{\lambda}}(0) \begin{pmatrix} 1 & 0 \\ 0 & e^{i\pi\Lambda} \end{pmatrix}\,, \\
\overline{w}_\mu(2\pi) &= \begin{pmatrix} 1 & 0 \\ 0 & e^{-i\pi\Lambda} \end{pmatrix} \overline{w}_\mu(0) \begin{pmatrix} 1 & 0 \\ 0 & e^{i\pi\Lambda} \end{pmatrix}\,, & \hat{\underline{w}}_\mu(2\pi) &= \begin{pmatrix} 1 & 0 \\ 0 & e^{-i\pi\Lambda} \end{pmatrix} \hat{\underline{w}}_\mu(0) \begin{pmatrix} 1 & 0 \\ 0 & e^{i\pi\Lambda} \end{pmatrix}\,.
\end{aligned}
\tag{6.8}
$$

We again see that these boundary conditions immediately show that the conserved BRST currents (4.12) are periodic and hence they are two time-independent BRST charges as follows from (4.15). In other words we have shown that classically the pure spinor string is well defined even in the case when the world-volume fields obey the twisted boundary conditions. On one hand the power of pure spinor formalism is that it allows to prove exact conformal invariance of the pure spinor string in $AdS_5 \times S^5$ background [48] and arguments given there crucially depend on the gauge invariance of the pure spinor string

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[4]For simplicity we restrict to the case of string with zero winding numbers $n_i = 0$.

with respect to subgroup $\mathrm{SO}(4,1) \times \mathrm{SO}(5)$. On the other hand the form of the action (3.47) explicitly depends on the coset representative that corresponds to the fixing of the gauge $\mathrm{SO}(4,1) \times \mathrm{SO}(5)$. Consequently the action (3.47) is not suitable for the analysis of the general properties of the pure spinor sigma model with twisted boundary conditions.

To find such a formulation we would like to express the theory where the fundamental fields obey the twisted boundary conditions in terms of the original currents J and ghost variables that appear in the action (3.1) and that obey some form of the twisted boundary conditions:

$$
\begin{aligned}
J(2\pi) &= N(\Lambda)J(0)N^{-1}(\Lambda) \,, \\
\lambda(2\pi) &= N(\Lambda)\lambda(0)N^{-1}(\Lambda) \,, \qquad \hat{\lambda}(2\pi) = N(\Lambda)\hat{\lambda}(0)N^{-1}(\Lambda) \,, \\
w_\mu(2\pi) &= N(\Lambda)w_\mu(0)N^{-1}(\Lambda) \,, \quad \hat{w}_\mu(2\pi) = N(\Lambda)\hat{w}_\mu(0)N^{-1}(\Lambda) \,,
\end{aligned}
\tag{6.9}
$$

for some matrix N that depends on Λ only. If we were able to find such a formulation then we would get the original action (3.1) with the explicit gauge invariance but where now the world-volume fields obey the twisted boundary conditions (6.9). Since the algebraic renormalisation arguments given [48] (see also [54] are sensitive to the UV properties of the theory we could then argue that (3.1) with fields obeying the twisted boundary conditions (6.9) defines exact quantum field theory. It turns out however that it is not possible to find such a form of the boundary conditions.

To be more precise we try to find the boundary conditions of the original currents $J^{(i)}, i = 1, 2, 3$ and ghosts $\lambda, \hat{\lambda}$ using the relations (3.38) and (3.46). These relations imply that $J, \lambda, \hat{\lambda}$ explicitly depend on g that has the form

$$
g = \begin{pmatrix} \tilde{g}_a & 0 \\ 0 & \tilde{g}_s \end{pmatrix}
\tag{6.10}
$$

where $\tilde{g}_a(2\pi) = \tilde{g}(0)$ as follows from the fact that g_a parametrises AdS_5. More difficult problem is to find the boundary condition for g_s. Recall that this group element has the form

$$
\tilde{g}_s = \exp\left(\frac{i}{2} y_a \Gamma_a \right) \,,
\tag{6.11}
$$

where y_a parametrise the five-sphere and $\Gamma_a, a = 1, \dots, 5$ are the Dirac matrices for $\mathrm{SO}(5)$. The variables y_a are related to ρ, ϕ given in (5.4) as

$$
\begin{aligned}
y_1 &= \frac{1}{2}\frac{|y|}{\sin|y|}(u_1 - u_1^*) \,, \quad y_2 = -\frac{1}{2}\frac{|y|}{\sin|y|}(u_2 + u_2^*) \,, \\
y_3 &= \frac{1}{2}\frac{|y|}{\sin|y|}(u_2 - u_2^*) \,, \quad y_4 = -\frac{1}{2}\frac{|y|}{\sin|y|}(u_1 + u_1^*) \,, \\
y_5 &= \frac{1}{2}\frac{|y|}{\sin|y|}(u_3 - u_3^*) \,, \quad |y| = -\sin^{-1}\left(\frac{u_3 + u_3^*}{2}\right) \,, \\
|y|^2 &= y_1^2 + y_2^2 + y_3^2 + y_4^2 + y_5^2 \,.
\end{aligned}
\tag{6.12}
$$

Then using the fact that g_s obeys following boundary conditions

$$
g_s(2\pi) = e^{-i\pi\Lambda} g_s(0) e^{-i\pi\Lambda}
\tag{6.13}
$$

we easily find the twisted boundary conditions for u_i

$$u_1(2\pi) = e^{-i\pi(\Lambda_1+\Lambda_3)}u_1(0) \,,$$
$$u_2(2\pi) = e^{-i\pi(\Lambda_1+\Lambda_4)}u_2(0) \,,$$
$$u_3(2\pi) = e^{-i\pi(\Lambda_1+\Lambda_2)}u_3(0) \,. \tag{6.14}$$

Using the boundary conditions for u_i given above we can easily determine the boundary condition for $|y|$ given in (6.12)

$$|y|(2\pi) = \sin^{-1}\left[2\sin|y|(0)\cos(\pi(\Lambda_1+\Lambda_2)) + 2i\sin(\pi(\Lambda_1+\Lambda_2)\frac{y_5(0)}{|y|(0)}\right] \,. \tag{6.15}$$

This result clearly demonstrates that it is not possible to find the appropriate N matrix introduced in (6.9) for general values of Λ. This is a consequence of the fact that the global symmetries of the coset are realised non-linearly on coordinates y.

Since we have shown that is not possible find the matrix N for general Λ it turns out that the arguments in [48] that were based on the existence of local gauge symmetry cannot be applied to the currents that obey twisted boundary conditions. However, looking at the above equations we see that for $\nu_i \in Z$, where ν_i is defined as

$$\nu_i = \epsilon_{ijk}\gamma_j J_k \tag{6.16}$$

the world-sheet fields in the original $AdS_5 \times S^5$ background obey the periodic boundary conditions as well. It is interesting to compare this requirement with the analysis performed in [25] where the significance of the solution with integer ν_i was stressed from different point of view. We are not going to perform the same analysis since we have not studied the classical solutions of the pure spinor string in the original $AdS_5 \times S^5$ background however we would like to stress some interesting points considering the condition that ν_i is an integer. For $\nu_i \neq 0$ we have consistent string dynamics if γ_i are rational since J_i take integer values in quantum theory. Secondly, the condition $\nu_i = 0$ has the general solution

$$\nu_i = 0 : \quad J_i = c\gamma_i \,. \tag{6.17}$$

Since again J_i have to be integer in quantum theory these solutions exist for special values of γ_i.

Returning back to (6.12) we see that the matrix g_s is periodic. Then using also the fact that $\mathbf{J}^{(i)}$ are periodic as well we obtain that the currents J given in (3.1) obey the standard boundary conditions. In other words we can formulate the dynamics of the theory in terms of the original currents J and the action (3.1) is manifestly gauge invariant. According to the analysis given in [48] the pure spinor string action in $AdS_5 \times S^5$ possesses quantum BRST invariance and also exact conformal invariance. Then using the TsT transformations we can map the pure spinor string in the γ_i-deformed background that obey the conditions that ν_i-that define ground states of the pure spinor world-sheet quantum field theory in γ_i-deformed background- are integer to the the pure spinor string in $AdS_5 \times S^5$ with periodic boundary conditions. But since the pure spinor string in the $AdS_5 \times S^5$ with periodic

boundary conditions possesses exact quantum conformal invariance we can expect that the world-sheet theory of the pure spinor string in γ_i-deformed background formulated above the ground states with ν_i integer is well defined quantum field theory.

Since we found that the proof of the conformal invariance of [48] strictly depends on the manifest isometries of the background, in other — more realistic — situations (for instance, N=1 supersymmetries backgrounds) another way of proving the conformal invariance has to be developed. This would be very important in order to prove conformal invariance of the theory expanded around other non trivial solutions which cannot be related to the undeformed original theory. We would like to stress that this is a subject of present investigation since it is not clear at the moment if all solution of Green-Schwarz formulation of superstring on $AdS_5 \times S^5$ string theory with a given gauge fixing are equivalent to pure spinor formulation [69].

7. Lax pair for twisted pure spinor string

Our goal is to find, using the relations (2.22) the Lax pair for string in TsT transformed background if an isometry invariant Lax pair for pure spinor string in flat background is known. An existence of Lax pair in deformed theory strongly supports classical integrability of the theory [53, 58–62, 68, 70].

We begin with the recalling the structure of Lax pair for pure spinor string in $AdS_5 \times S^5$. In the covariant pure spinor formalism the problem has been studied in [58]. It was shown here that there exists set of left-invariant currents $\hat{J}(u)$ [5]

$$
\begin{aligned}
\hat{J}_\mu(u) = J_\mu &+ (\eta_{\mu\nu}(\cosh u - 1) + \epsilon_{\mu\nu}\sinh u)J^{\nu(2)} + \\
&+ (\eta_{\mu\nu}(\cosh u e^{u/2} - 1) + \epsilon_{\mu\nu}\sinh u e^{u/2})J^{\nu(1)} + \\
&+ (\eta_{\mu\nu}(\cosh u e^{-u/2} - 1) + \epsilon_{\mu\nu}\sinh u e^{-u/2})J^{\nu(3)} + \\
&+ \sinh u e^u \tilde{\mathcal{P}}_{\mu\nu}N^\nu - \sinh u e^{-u}\mathcal{P}_{\mu\nu}\hat{N}^\nu
\end{aligned}
\tag{7.1}
$$

that satisfy the flatness condition

$$
d\hat{J} + \hat{J} \wedge \hat{J} = 0
\tag{7.2}
$$

that is a consequence of the equations of motion for J and ghost fields and also of the flatness of J. Note also that $\hat{J}$ obeys the the 'initial' condition $\hat{J}(0) = J$.

The Lax connection given above cannot be used to derive the Lax connection in deformed background since $J_\mu^{(i)}$ given there explicitly depend on ϕ and consequently $\hat{J}_\mu$ is not isometry invariant. Moreover, if we express $J^{(0)}$ using (3.28) it turns out that it explicitly depends on $g(z)$. This is due the lack of covariance under the gauge symmetry $\mathrm{SO}(4,1) \times \mathrm{SO}(5)$ of the currents hJ_μ after the TsT transformations are performed. In order to solve this problem we construct a covariant version of the currents by changing

---

[5]Our spectral parameter u is related to the spectral parameter μ of [58] by $\mu = e^u$. Note also that we have chosen one particular solution from the ones found in [58] in order to obey the initial condition $\hat{J}_\mu(0) = J_\mu$. It is remarkable that the classical theory admits the same two one-parameter families of flat currents if one sets the contribution of the pure spinor ghost N to zero.

the current in a similar way as [2, 3, 61, 64]. The corresponding flatness conditions will be manifestly covariant if the differential d is substituted by a covariant differential.

Let us write the flat current $\hat{J}$ as

$$\hat{J} = g^{-1}(z)dg(z) + g^{-1}(z)\hat{J}'g(z) . \tag{7.3}$$

Then the flatness of $\hat{J}$ implies

$$d\hat{J} + \hat{J} \wedge \hat{J} = g^{-1}(d\hat{J}' + \hat{J}' \wedge \hat{J}')g = 0 \tag{7.4}$$

and hence $d\hat{J}' + \hat{J}' \wedge \hat{J}' = 0$. Now (7.1) implies

$$
\begin{aligned}
\hat{J}'_\mu(u) &= g(z)\hat{J}_\mu g^{-1}(z) - \partial_\mu g(z)g^{-1}(z) \\
&= \mathbf{J}_\mu + [(\eta_{\mu\rho}(\cosh u - 1) + \epsilon_{\mu\rho}\sinh u)\eta^{\rho\sigma}\mathbf{J}^{(2)}_\sigma + \\
&\quad + (\eta_{\mu\rho}(\cosh ue^{u/2} - 1) + \epsilon_{\mu\rho}\sinh ue^{u/2})\eta^{\rho\sigma}\mathbf{J}^{(1)}_\sigma + \\
&\quad + (\eta_{\mu\rho}(\cosh ue^{-u/2} - 1) + \epsilon_{\mu\rho}\sinh ue^{-u/2})\eta^{\rho\sigma}\mathbf{J}^{(3)}_\sigma + \\
&\quad + \sinh ue^u\tilde{\mathcal{P}}_{\mu\nu}\overline{N}^\nu - \sinh ue^{-u}\mathcal{P}_{\mu\nu}\underline{\hat{N}}^\nu] ,
\end{aligned}
\tag{7.5}
$$

where we have also used (3.38) and (3.46). The Lax connection $\hat{J}'$ (7.5) still has explicit dependence on ϕ^i but this can be easily eliminated using the factorisation property of G and redefinition of the fermions and pure spinors. Explicitly, using the relations (5.13), (5.15) and (5.16)) we can write (7.5) as

$$
\begin{aligned}
\hat{J}'_\mu &= M\tilde{\mathbf{J}}_\mu M^{-1} + M[(\eta_{\mu\rho}(\cosh u - 1) + \epsilon_{\mu\rho}\sinh u)\eta^{\rho\sigma}\tilde{\mathbf{J}}^{(2)}_\sigma + \\
&\quad + (\eta_{\mu\rho}(\cosh ue^{u/2} - 1) + \epsilon_{\mu\rho}\sinh ue^{u/2})\eta^{\rho\sigma}\tilde{\mathbf{J}}^{(1)}_\sigma + \\
&\quad + (\eta_{\mu\rho}(\cosh ue^{-u/2} - 1) + \epsilon_{\mu\rho}\sinh ue^{-u/2})\eta^{\rho\sigma}\tilde{\mathbf{J}}^{(3)}_\sigma + \\
&\quad + \sinh ue^u\tilde{\mathcal{P}}_{\mu\nu}\tilde{N}^\nu - \sinh ue^{-u}\mathcal{P}_{\mu\nu}\tilde{\hat{N}}^\nu]M^{-1} \equiv M(\hat{\mathbf{J}}_\mu - \frac{i}{2}\partial_\mu\Phi)M^{-1} .
\end{aligned}
\tag{7.6}
$$

Then the flatness condition for $\hat{J}'$ implies

$$d\hat{J}' + \hat{J}' \wedge \hat{J}' = M(d\hat{\mathbf{J}} + \hat{\mathbf{J}} \wedge \hat{\mathbf{J}})M^{-1} = 0 \tag{7.7}$$

and consequently we obtain the flatness condition for $\hat{\mathbf{J}}$

$$d\hat{\mathbf{J}} + \hat{\mathbf{J}} \wedge \hat{\mathbf{J}} = 0 . \tag{7.8}$$

We see, following the arguments given in section (5) that the Lax connection $\hat{\mathbf{J}}$ depends on the derivatives of Φ only. Then following the arguments given in [2] we can determine the Lax connection for pure spinor strings in the γ-deformed $AdS_5 \times S^5$ when we express $\partial_\mu\phi_i$ in terms of $\partial_\mu\tilde{\phi}_i$ with the help of the relations (2.22) and also using the fact that $\hat{\mathbf{J}}$ depends on variables that are neutral under $\mathrm{U}(1)$ only. By construction the Lax connection $\hat{\mathbf{J}}$ is flat, it is invariant under $\mathrm{U}(1)$ isometries and it also obeys the periodic boundary conditions. It can be used to compute the monodromy matrix $T(u)$ that is defined as the path-ordered exponential of the spatial component of $\hat{\mathbf{J}}_\sigma$

$$T(u) = P\exp \int_0^{2\pi} d\sigma\hat{\mathbf{J}}_\sigma(u) . \tag{7.9}$$

On the other hand we have argued that in order to study the quantum properties of the string theory in TsT-deformed background it is necessary that the world-sheet modes in the original $AdS_5 \times S^5$ background obey the periodic boundary conditions. This results also implies that $\hat{J}'$ and $\hat{J}$ are periodic as well and their analysis can be performed as in [48, 53].

Acknowledgments

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